PLAY-THE-WINNER SAMPLING IN SELECTING THE BETTER OF TWO BINOMIAL POPULATIONS*

by

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Technical Report No. 147

University of Minnesota
Minneapolis, Minnesota

November, 1970

*This research was supported by National Science Foundation Grant GP-11021.

I dedicate this thesis

to

Linda

Acknowledgement

I would like to thank Professor Milton Sobel for his expert guidance and encouragement in the writing of this thesis. It has been a pleasure to work under him. It also has been an experience that I have found invaluable and one that I will never forget.

I would like to thank Lynn Zmistowski, the secretary of the Statistics Department, for her many efforts on my behalf during the writing of this thesis.

I would like to thank Gerry DuChaine for his tireless efforts in the original typing and for the subsequent corrections. He is an excellent statistical typist.

I would like to thank Leo May for the programming that resulted in all the tables of this thesis. He worked long and hard and always did an excellent job.

Finally, I thank the professors and students in the Department of Statistics for making this thesis possible.

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ABSTRACT

Play-the-Winner Sampling in Selecting the Better of Two Binomial Populations

The better of two binomial populations A and B (also, referred to as treatments) is defined to be the one with the higher probability of success. A correct selection is made when the better treatment is correctly identified. Let treatment 1 (resp., 2) with probability p_1 (resp., p_2) of success on a single trial refer to the better (resp., poorer) of the two treatments, so that, allowing equality, we have $p_1 \geq p_2$.

One particular application of the problem is the medical problem of determining which of two treatments is better for a given disease.

Based on the Sobel-Weiss formulation for the problem of comparing two or more binomial populations, a comparison is made between the PW (play-the-winner) and VT (vector-at-a-time) sampling rules for the fixed sample size selection problem within the framework of the usual two-decision approach, the two decisions being (1) treatment A is the better treatment and (2) treatment B is the better treatment. It is found that for any (even) total number of observations, the probability of correct selection for the PW and VT sampling rules are identical. It is then shown that the expected number of observations on the poorer treatment is less for PW sampling than for VT sampling.

PW and VT sampling are then compared for the three-decision problem, where the three decisions are (1) treatment A is better, (2) for all practical purposes neither treatment is better and (3) treatment B is better. The test considered is based on the statistic $W = \frac{S_A - S_B}{N}$, where N is the total number of observations. Decision (1) is made

if $W > f$, decision (2) is made if $-f \leq W \leq f$ and decision (3) is made if $W < -f$. The constants f and N are determined so that (a) the probability of correctly identifying the better treatment for $p_1 - p_2 \geq \Delta^*$ is at least P_1^* and (b) the probability of decision (2) for $p_1 = p_2$ is at least P_2^*, and (c) N is as small as possible; here P_1^*, P_2^* and Δ^* are preassigned constants. The restriction $p_1 \leq \theta < 1$, with θ also preassigned, is placed on the problem for the PW rule since the PW rule cannot satisfy the above conditions (b) without it. It is shown that the total number of observations is less for VT than for PW sampling for all θ considered.

For the two-decision problem with the inverse sampling termination rule, a class of procedures is considered in which the s^{th} procedure, $R_I^{(s)}$, switches only after s successive failures. It is found that by taking $s = 1$ (this is the PW sampling rule), the expected number of observations until termination evaluated at $p_1 = p_2$ is minimized. In this sense, the PW rule is optimal for this class of procedures.

Finally, two open questions in this area of research are considered. The first question concerns the extension of the present theory to sampling in blocks of a fixed size. The second one concerns an extension to the trinomial problem, where a treatment can achieve one of three (rather than two) possible results on a single trial.

Milton Sobel

CHAPTER I

Introduction and Summary

1.1 Introduction.

The 2-armed bandit problem was first introduced by Robbins [9]. Although not primarily a selection problem, it led to a formulation (by Sobel and Weiss) of the problem of selecting the better of two binomial populations, upon which this thesis is based.

The 2-armed bandit problem concerns two binomial populations, each having two outcomes called success and failure, with a fixed but unknown probability of success. A series of independent trials is conducted and it is up to the experimenter to decide on any given trial, from which of the two binomial populations to sample; the object is to maximize the expected proportion of successes as the number of trials becomes infinite. Different sampling rules are then considered. Among them is the PW rule (defined at the beginning of Section 2.2). Research along these lines was conducted by Isbell [5], Bradt, Johnson and Karlin [2], Feldman [3], Smith and Pyke [11] and Samuels [10].

Sobel and Weiss drastically change the nature of the problem and place it into the framework of ranking and selection. In [13], [14] and [15], they use a formulation explored in some detail in a monograph by Bechhofer, Kiefer and Sobel [1]. The two binomial populations are now referred to as treatments, called 1 and 2. Treatments 1 and 2 have probability of success p_1 and p_2, respectively; it is assumed that $p_1 \geq p_2$, thus making 1 the "better" treatment. For convenience, we refer to treatments 1 and

2 when the corresponding p's are ordered and to treatments A and B as the given treatments of the problem without ordered p-values . With probability one, sampling is to be terminated and a decision as to the better treatment is to be made.(This is in direct contrast to the 2-armed bandit problem, where sampling goes on forever.) A correct selection (CS) corresponds to identifying 1 as the better treatment, at the termination of sampling. (In the 2-armed bandit problem, no decision is ever made as to the better treatment.) Letting $\Delta = p_1 - p_2$, only those procedures, R, are considered, which satisfy the requirement on the probability of a correct selection CS that

$$(1.1.1) \qquad P\{CS|R\} \geq P^* \quad \text{whenever} \quad \Delta \geq \Delta^*,$$

where $0 < \Delta^* < 1$ and $\frac{1}{2} < P^* < 1$ are preassigned constants. Comparisons of (1) the expected number of observations until termination, (2) the expected number of observations on the poorer treatment at termination or (3) the expected value of any appropriate loss function, are made to decide between procedures satisfying (1.1.1).

One particular application of the problem is the medical problem of determining which of two treatments is better for a given disease. Here, we assume that the results of any trial are immediately obtained and can be used to determine which treatment to use on the very next trial.

Sobel and Weiss consider the PW and VT sampling rules (defined at the beginning of Sections 2.2 and 2.3) and compare the two of them, under different termination rules. Several such termination rules are considered. The first rule depends on computing $|S_1 - S_2|$

at each stage, where S_1 and S_2 are the numbers of successes of treatments 1 and 2, respectively; termination occurs when $|S_1 - S_2| = r$. A second stopping rule corresponds to inverse sampling, where termination occurs when S_1 or S_2 reaches r' successes. r and r' are determined from (1.1.1). Finally, a truncated sequential procedure has been described by Kiefer and Weiss [6], for the VT rule. A decision is to be made at or before test N, where the termination rule requires that $|S_1 - S_2| = s$. A survey of the research in this area, both completed and currently in progress, is undertaken by Sobel and Weiss in [16].

1.2 Summary.

This thesis, as mentioned before, is based on the Sobel and Weiss formulation for the problem of comparing two or more binomial populations. The first part of Chapter II is concerned with the comparison of the PW and VT sampling rules for the fixed sample size selection problem within the framework of the usual two-decision approach, the two decisions being that (1) treatment A is the better treatment and (2) treatment B is the better treatment. It is found that for any (even) total number of observations, the probability of correct selection for the PW and VT sampling rules are identical; the VT rule requires an even number if randomization is not used. A similar result is found under inverse sampling in [14]. It is then shown that the expected number of observations on the poorer treatment is less for PW than for VT sampling for all pairs (p_1,p_2) with $p_2 < p_1$.

In the second part of Chapter II, PW and VT sampling are compared for the three-decision problem, where the three decisions are that

(d_1) treatment A is better, (d_2) for all practical purposes neither treatment is better, and (d_3) treatment B is better. The test considered is based on the statistic $W = \frac{S_A - S_B}{N}$, where N is the total number of observations; decision (d_1) is made if $W > f$, decision (d_2) is made if $-f \leq W \leq f$ and decision (d_3) is made if $W < -f$. Letting p_α denote the p-value for treatment $\alpha (\alpha = A, B)$, the values of N and f are determined so that

$$\text{(i)} \quad P\{d_1\} \geq P_1^* \qquad \text{for } p_1 = p_A \text{ and } p_1 - p_2 \geq \Delta^*$$

$$\text{(ii)} \quad P\{d_2\} \geq P_2^* \qquad \text{for } p_1 = p_2 \leq \theta$$

$$\text{(iii)} \quad P\{d_3\} \geq P_3^* \qquad \text{for } p_1 = p_B \text{ and } p_1 - p_2 \geq \Delta^*,$$

where P_1^*, P_2^*, Δ^* and θ are all preassigned. In addition (for $\Delta^* \to 0$) if (N_0, f_0) is any other pair satisfying these conditions then $N_0 \geq N$. For the PW rule $\frac{1 + \Delta^*}{2} \leq \theta < 1$ and for the VT rule, $\theta = 1$. An important result obtained is that regardless of the value of θ used in (ii) (for the PW rule) the number of observations N_{PW} needed for the PW-sampling rule is not less than the number N_{VT} needed for the VT-sampling rule.

Let $W_{12} = \frac{S_1 - S_2}{N}$. Although W_{12} is not a statistic (since it is not observable) we can use it to simplify our probability requirement. We treat the two kinds of correct selections (with treatment A and with treatment B) symmetrically using the same Δ^* in (i) and (iii) above and taking $P_1^* = P_3^*$. We note that (i) $W_{12} > f$ means that either $W > f$ and $p_1 = p_A$ or $W < -f$ and $p_1 = p_B$ and (ii) $-f \leq W_{12} \leq f$ is equivalent to the same inequalities on W. As a result of this we can use W_{12} and write the three requirements above as two in the form

(i) $P\{CS\} = P\{W_{12} > f\} \geq P_1^*$ for $p_1 - p_2 \geq \Delta^*$

(ii) $P\{CD\} = P\{-f \leq W_{12} \leq f\} \geq P_2^*$ for $p_1 = p_2 \leq \theta$

where CD stands for correct decision.

In Chapter III, we return to the two-decision problem, but under the inverse sampling termination rule. A class of inverse sampling procedures is considered, in which the s^{th} procedure, $R_I^{(s)}$, switches only after s successive failures. It is found that the best ranking and selection results are obtained by taking $s = 1$ (the PW-sampling rule). Specifically, the expected number of observations until termination, under procedure $R_I^{(s)}$, is (1) maximized for all s, when $\Delta = 0$ and (2) minimized at $s = 1$, for $P^* \to 1$, $\Delta^* \to 0$ and $\Delta = 0$.

Finally, in Chapter IV, we discuss two open questions in this area of research, for which the author of this thesis has obtained only partial results.

CHAPTER II

PW vs. VT Sampling for the Fixed Sample Size Selection Problem

2.1 Introduction.

In this chapter, we study the effect of the PW sampling rule on the fixed sample size selection problem, where a fixed total number N of observations is observed from both populations. We define a procedure, denoted by R_N, based on the PW sampling rule and compare it for large N to a procedure which entails observing an equal number of observations from both populations.

2.2 Exact Results for Procedure R_N and a Related Procedure R_N'.

We first describe the aforementioned procedure R_N based on a fixed total sample size N as follows. At the outset, we select one population by randomizing between the two of them (with equal probabilities for each) and use it for the first observation; then we follow the so-called play-the-winner (PW) sampling rule, i.e., continue the same treatment if we get a success and switch to the other treatment if we get a failure. After a total of N observations, we terminate and select the population with the greater number of successes as the better one. In case of a tie at termination, we select one by randomizing between the two populations (with equal probabilities for each).

The size N of procedure R_N is determined so that the $P\{CS|R_N\}$ satisfies (1.1.1).

Let treatments 1 and 2 have the properties given in Section 1.1. Let S_1 (resp., S_2) denote the current number of successes with treatment 1 (resp., 2). Let M denote the number of observations that have yet to be taken to reach N. Let S_{1M} (resp., S_{2M}) denote the number of successes with treatment 1 (resp., 2) when there are M observations still to be taken to reach N. Letting

$$(2.2.1)\qquad \begin{aligned} L_{M,d} &= P\{CS \mid S_{1M} - S_{2M} = d \text{ and the next treatment is } 1\}, \\ T_{M,d} &= P\{CS \mid S_{1M} - S_{2M} = d \text{ and the next treatment is } 2\}, \end{aligned}$$

and $q_i = 1 - p_i$ $(i = 1, 2)$, the recursive relations for procedure R_N are

$$(2.2.2)\qquad \begin{aligned} L_{M,d} &= p_1 L_{M-1,d+1} + q_1 T_{M-1,d}, \\ T_{M,d} &= p_2 T_{M-1,d-1} + q_2 L_{M-1,d}; \end{aligned}$$

the boundary conditions are given by

$$(2.2.3)\qquad L_{0,d} = T_{0,d} = \begin{cases} 1 & \text{for } d > 0 \\ \frac{1}{2} & \text{for } d = 0 \\ 0 & \text{for } d < 0 \end{cases} .$$

We define the generating functions

$$(2.2.4)\qquad U = \sum_{M=1}^{\infty} \sum_{d=-\infty}^{+\infty} L_{M,d} x^M y^d, \quad V = \sum_{M=1}^{\infty} \sum_{d=-\infty}^{d=+\infty} T_{M,d} x^M y^d .$$

Multiplying each of the terms in (2.2.2) by $x^M y^d$, summing over M and d and using the boundary conditions, we easily obtain

$$(2.2.5)\qquad \begin{aligned} U(1-p_1 \tfrac{x}{y}) &= q_1 xV + (p_1 \tfrac{x}{y} + q_1 x)(\tfrac{y}{1-y} + \tfrac{1}{2}), \\ V(1-p_2 xy) &= q_2 xU + (p_2 xy + q_2 x)(\tfrac{y}{1-y} + \tfrac{1}{2}). \end{aligned}$$

Solving (2.2.5) for U and V results in

$$(2.2.6)\qquad U = \left[\frac{1-p_2xy+q_1x}{D} - 1\right]\left[\frac{y}{1-y} + \frac{1}{2}\right],$$

$$V = \left[\frac{q_2x}{1-p_2xy}\left(\frac{1-p_2xy+q_1x}{D} - 1\right) + \frac{p_2xy+q_2x}{1-p_2xy}\right]\left[\frac{y}{1-y} + \frac{1}{2}\right],$$

where D is defined by

$$(2.2.7)\qquad D^{-1} = [(1-p_1\tfrac{x}{y})(1-p_2xy) - q_1q_2x^2]^{-1} = \sum_{i=0}^{\infty}\frac{(q_1q_2x^2)^i}{(1-p_1\frac{x}{y})^{i+1}(1-p_2xy)^{i+1}}$$

$$= \sum_{i=0}^{\infty}(q_1q_2x^2)^i \sum_{j=0}^{\infty}\binom{i+j}{j}(p_1\tfrac{x}{y})^j \sum_{k=0}^{\infty}\binom{i+k}{k}(p_2xy)^k.$$

As a result of the randomization at the outset,

$$(2.2.8)\qquad P\{CS|R_N\} = \tfrac{1}{2}(L_{N,0} + T_{N,0})$$

and hence the desired result is the coefficient of x^Ny^0 in $(U + V)/2$. With this in mind, we first use (2.2.7) to make some preliminary calculations. These calculations are more extensive than necessary for solving (2.2.8) because they will also be used for obtaining equations (2.2.33) and (2.2.34) later in the section.

We begin by finding the coefficient $C_{N,0}^{(\alpha,\beta)}$ of x^Ny^0 in

$$(2.2.9)\qquad \frac{x^\alpha y^\beta}{D}\,\frac{y}{1-y}\,(1-p_2xy)^{1-\alpha} = x^\alpha y^\beta\left(\frac{y}{1-y}\right)\sum_{i=0}^{\infty}\frac{(q_1q_2x^2)^i}{(1-p_1\frac{x}{y})^{i+1}(1-p_2xy)^{\alpha+i}}$$

$$= \sum_{t=1}^{\infty}\sum_{i=0}^{\infty}\sum_{k=0}^{\infty}\sum_{h=0}^{\infty} s_{i,k}\,x^{2i+\alpha+h+k}y^{\beta+t+k-h},$$

where

$$(2.2.10)\qquad s_{i,k} = (q_1q_2)^i\binom{i+h}{h}\binom{i+\alpha-1+k}{k}p_1^hp_2^k,$$

and as a result of setting the exponents of x and y, respectively, equal to N and 0, the sums on t and h are removed by setting

$$t = N - 2k - 2i - \alpha - \beta \geq 1, \; h = N - k - 2i - \alpha \geq 0. \tag{2.2.11}$$

We then sum $s_{i,k}$ over i and k restricted by the condition that $t \geq 1$ or $i + k \leq (N-1-\alpha-\beta)/2$. Writing j for i + k and defining $\binom{x}{0} = 1$ for $x \geq 0$, yields the desired result

$$C_{N,0}^{(\alpha,\beta)} = p_1^{N-\alpha} \sum_{i=0}^{H(\alpha,\beta)} \left(\frac{q_1 q_2}{p_1 p_2}\right)^i \sum_{j=i}^{H(\alpha,\beta)} \binom{N-j-\alpha}{i} \binom{j+\alpha-1}{i+\alpha-1} \left(\frac{p_2}{p_1}\right)^j, \tag{2.2.12}$$

where $H(\alpha, \beta)$ is the largest integer less than or equal to $\frac{N-1-\alpha-\beta}{2}$.

The pairs (α, β) in (2.2.12) that we will use are

$$(\alpha, \beta) = \begin{cases} (1, -1) \\ (1, 0) \\ (1, 1) \\ (2, 0) \end{cases} . \tag{2.2.13}$$

Similarly, $B_{N,0}^{(\alpha,\ \beta)}$, the coefficient of $\frac{x^\alpha y^\beta}{D} \frac{y}{1-y} (1-p_2 xy)$, is given by

$$B_{N,0}^{(\alpha,\beta)} = p_1^{N-\alpha} [\eta_{1+\alpha+\beta,N} + \sum_{i=1}^{H(\alpha,\beta)} \left(\frac{q_1 q_2}{p_1 p_2}\right)^i \sum_{j=i}^{H(\alpha,\beta)} \binom{N-j-\alpha}{i} \binom{j-1}{i-1} \left(\frac{p_2}{p_1}\right)^j], \tag{2.2.14}$$

where $H(\alpha, \beta)$ is defined above and where

$$\eta_{1+\alpha+\beta,N} = \begin{cases} 0 & \text{if } N < 1 + \alpha + \beta \\ 1 & \text{if } N \geq 1 + \alpha + \beta \end{cases} ; \tag{2.2.15}$$

we shall use this for the pairs (α, β) such that

$$(\alpha, \beta) = \begin{cases} (0, 0) \\ (1, -1) \\ (1, 0) \\ (1, 1) \\ (2, 0) \end{cases} . \tag{2.2.16}$$

In particular, let us note that

$$(2.2.17)\qquad B_{N,0}^{(\alpha,\beta)} = C_{N,0}^{(\alpha,\beta)} = \begin{cases} p_1^{N-\alpha} & \text{for } N = 1+\alpha+\beta \\ 0 & \text{for } N < 1+\alpha+\beta \end{cases}$$

and that β enters (2.2.12) and (2.2.14) only through $H(\alpha, \beta)$.

Needed also is the coefficient $A_{N,0}^{(\alpha,\beta)}$ of

$$(2.2.18)\qquad \frac{x^\alpha y^\beta}{D}(1-p_2xy)^{1-\alpha} = \sum_{i=0}^{\infty}\sum_{h=0}^{\infty}\sum_{k=0}^{\infty} s_i x^{2i+\alpha+h+k} y^{\beta+k-h},$$

where s_i is again given by (2.2.10). Here we write s_i since as a result of setting the exponents of x and y, respectively, equal to N and 0, the sums on h and k are removed by setting

$$(2.2.19)\qquad h = \frac{N-\alpha+\beta}{2} - i \geq 0,\ k = \frac{N-\alpha-\beta}{2} - i \geq 0.$$

We then sum s_i over i restricted by the condition that $i \leq \frac{N-\alpha-\beta}{2}$. Define $\binom{x}{y} = 0$ if $x < y$ or if either x or y is a fraction or is negative. Then $A_{N,0}^{(\alpha,\beta)}$ is given by

$$(2.2.20)\qquad A_{N,0}^{(\alpha,\beta)} = (p_1p_2)^{\frac{N-\alpha}{2}}\left(\frac{p_1}{p_2}\right)^{\frac{\beta}{2}} \sum_{i=0}^{H'(\alpha,\beta)} \left(\frac{q_1q_2}{p_1p_2}\right)^i \binom{\frac{N-\alpha+\beta}{2}}{i}\binom{\frac{N+\alpha-\beta}{2}-1}{i+\alpha-1},$$

where $H'(\alpha, \beta)$ is the largest integer less than or equal to $\frac{N-\alpha-\beta}{2}$. We use the pairs (α, β) given in (2.2.13). In particular, we note that $A_{N,0}^{(\alpha,\beta)} = 0$ if N has a parity different from that of $\alpha - \beta$ (the latter having the same parity as $\alpha + \beta$) or if $N < \alpha + \beta$. Let us also note that

$$(2.2.21)\qquad A_{N,0}^{(\alpha,\beta)} = \begin{cases} p_1^\beta & \text{if } N = \alpha+\beta \text{ and } \beta \geq 0 \\ 0 & \text{if } N = \alpha+\beta \text{ and } \beta = -1 \end{cases}.$$

Similarly the coefficient $D_{N,0}^{(\alpha,\beta)}$ of $\frac{x^\alpha y^\beta}{D}(1-p_2xy)$ for $H'(\alpha, \beta) \geq 0$ is given by

$$(2.2.22)\qquad D_{N,0}^{(\alpha,\beta)} = \{(p_1p_2)^{\frac{N-\alpha}{2}}(\frac{p_1}{p_2})^{\frac{\beta}{2}} \sum_{i=1}^{H'(\alpha,\beta)} (\frac{q_1q_2}{p_1p_2})^i \binom{\frac{N-\alpha+\beta}{2}}{i}\binom{\frac{N-\alpha-\beta}{2}-1}{i-1} + p_1^{\beta}\delta_{\alpha+\beta,N}$$

where

$$(2.2.23)\qquad \delta_{\alpha+\beta,N} = \begin{cases} 0 & \text{if } N \neq \alpha+\beta \text{ or } \beta < 0 \\ 1 & \text{if } N = \alpha+\beta \text{ and } \beta \geq 0 \end{cases} .$$

Here, we need the pairs (α, β) given in (2.2.16). Note that

$$(2.2.24)\qquad D_{N,0}^{(\alpha,\beta)} = p_1^{\beta} \quad \text{for } N = \alpha+\beta,\ \beta \geq 0$$

and the first line of (2.2.22) gives zero for $N \leq \max(\alpha+\beta, \alpha-\beta)$.

Using (2.2.6), we find that, for $N > 0$, the coefficients of $x^N y^0$ in U and V, respectively, are given by

$$(2.2.25)\qquad L_{N,0} = B_{N,0}^{(0,0)} + q_1 C_{N,0}^{(1,0)} + \frac{1}{2} D_{N,0}^{(0,0)} + \frac{q_1}{2} A_{N,0}^{(1,0)} ,$$

$$T_{N,0} = q_2 C_{N,0}^{(1,0)} + q_1q_2 C_{N,0}^{(2,0)} + \frac{q_2}{2} A_{N,0}^{(1,0)} + \frac{q_1q_2}{2} A_{N,0}^{(2,0)} .$$

Making use of (2.2.8) and (2.2.25), we obtain

$$(2.2.26)\qquad P\{CS|R_N\} = \frac{1}{2} B_{N,0}^{(0,0)} + (\frac{q_1+q_2}{2}) C_{N,0}^{(1,0)} + (\frac{q_1q_2}{2}) C_{N,0}^{(2,0)}$$

$$+ \frac{1}{4} D_{N,0}^{(0,0)} + (\frac{q_1+q_2}{4}) A_{N,0}^{(1,0)} + (\frac{q_1q_2}{4}) A_{N,0}^{(2,0)} .$$

Substituting the values of $B_{N,0}^{(0,0)}$ etc., into (2.2.26) yields the desired exact expression

$$(2.2.27)\qquad P\{CS|R_N\} = \frac{1}{2} p_1^N [1 + \sum_{\alpha=0}^{2} p_1^{-\alpha}\psi_\alpha \sum_{i=0}^{H(\alpha,0)} (\frac{q_1q_2}{p_1p_2})^i \sum_{j=i}^{H(\alpha,0)} \binom{N-j-\alpha}{i}\binom{j+\alpha-1}{i+\alpha-1}(\frac{p_2}{p_1})^j]$$

$$+ \frac{1}{4} \sum_{\alpha=0}^{2} \psi_\alpha (p_1p_2)^{\frac{N-\alpha}{2}} \sum_{i=0}^{H'(\alpha,0)} (\frac{q_1q_2}{p_1p_2})^i \binom{\frac{N-\alpha}{2}}{i}\binom{\frac{N+\alpha-2}{2}}{i+\alpha-1},$$

where

$$(2.2.28)\qquad \psi_\alpha = \begin{cases} 1 & \text{for } \alpha = 0 \\ q_1 + q_2 & \text{for } \alpha = 1 \\ q_1 q_2 & \text{for } \alpha = 2 \end{cases} .$$

This exact expression (2.2.27) for the $P\{CS|R_N\}$ will be used later to show that the $P\{CS\}$ is the same for the PW rule and the VT rule for N even as mentioned in Section 1.2.

We next study a related procedure which we denote by R_N'. Let F_1 (resp., F_2) denote the current number of failures with treatment 1 (resp., 2). Define the score of 1 (resp., score of 2), denoted by C_1 (resp., C_2), to be equal to $S_1 - F_1$ (resp., $S_2 - F_2$). Let M denote the number of observations that have yet to be taken to reach N. Let C_{1M} (resp., C_{2M}) denote the value of the score associated with treatment 1 (resp., 2) when there are M observations still to be taken to reach N. We retain the same sampling and termination rules as in R_N. The difference is that, under R_N', at termination the treatment with higher score is declared to be better.

We use the same symbols $L_{M,d}$ and $T_{M,d}$ and define these as before except that the "$S_{1M} - S_{2M} = d$" in (2.2.1) are replaced by "$C_{1M} - C_{2M} = d$" for procedure R_N'. The recursion formulae for procedure R_N' are

$$(2.2.29)\qquad L_{M,d} = p_1 L_{M-1,d+1} + q_1 T_{M-1,d-1} ,$$

$$T_{M,d} = p_2 T_{M-1,d-1} + q_2 L_{M-1,d+1} ,$$

and the boundary conditions are

$$(2.2.30)\qquad L_{0,d} = T_{0,d} = \begin{cases} 1 & \text{for } d > 0 \\ \frac{1}{2} & \text{for } d = 0 \\ 0 & \text{for } d < 0 \end{cases} .$$

The required $P\{CS|R_N'\}$ is given by

$$(2.2.31) \quad P\{CS|R_N'\} = \tfrac{1}{2}(L_{N,0} + T_{N,0}).$$

By the same technique that was used for procedure R_N, we obtain from (2.2.29) and (2.2.30) that

$$(2.2.32) \quad V = \frac{q_2 \frac{x}{y} U + [p_2 xy + q_2 \frac{x}{y}][\frac{y}{1-y} + \frac{1}{2}]}{1 - p_2 xy},$$

$$U = \left[\frac{1 - p_2 xy + q_1 xy}{(1-p_2 xy)(1-p_1 \frac{x}{y}) - q_1 q_2 x^2} - 1\right]\left[\frac{y}{1-y} + \frac{1}{2}\right].$$

Let $C_{N,0}^{(\alpha,\beta)}$ etc., be defined as earlier in the section. Using (2.2.7), we then obtain for the coefficients of $x^N y^0$ in U and V, respectively,

$$(2.2.33) \quad L_{N,0} = B_{N,0}^{(0,0)} + q_1 C_{N,0}^{(1,1)} + \frac{1}{2} D_{N,0}^{(0,0)} + \frac{q_1}{2} A_{N,0}^{(1,1)},$$

$$T_{N,0} = q_2 C_{N,0}^{(1,-1)} + q_1 q_2 C_{N,0}^{(2,0)} + \frac{q_2}{2} A_{N,0}^{(1,-1)} + \frac{q_1 q_2}{2} A_{N,0}^{(2,0)}.$$

Making use of (2.2.31) and (2.2.33), we obtain

$$(2.2.34) \quad P\{CS|R_N'\} = \frac{B_{N,0}^{(0,0)}}{2} + \frac{q_2}{2} C_{N,0}^{(1,-1)} + \frac{q_1}{2} C_{N,0}^{(1,1)} + \frac{q_1 q_2}{2} C_{N,0}^{(2,0)} + \frac{D_{N,0}^{(0,0)}}{4} + \frac{q_2}{4} A_{N,0}^{(1,-1)} + \frac{q_1}{4} A_{N,0}^{(1,1)} + \frac{q_1 q_2}{4} A_{N,0}^{(2,0)}.$$

Substituting the values of $B_N^{(0,0)}$ etc., into (2.2.34), we obtain the desired exact expression

$$(2.2.35) \quad P\{CS|R_N'\} = \frac{1}{2} p_1^N \Big[1 + \sum_{(\alpha,\beta)\in L} p_1^{-\alpha} \psi_{\alpha,\beta} \sum_{i=0}^{H(\alpha,\beta)} \left(\frac{q_1 q_2}{p_1 p_2}\right)^i \cdot \sum_{j=i}^{H(\alpha,\beta)} \binom{N-j-\alpha}{i}\binom{j+\alpha-1}{i+\alpha-1}\left(\frac{p_2}{p_1}\right)^j\Big]$$

$$+\frac{1}{4}\sum_{(\alpha,\beta)\in L}(p_1p_2)^{\frac{N-\alpha}{2}}\left(\frac{p_1}{p_2}\right)^{\frac{\beta}{2}}\psi_{\alpha,\beta}\sum_{i=0}^{H'(\alpha,\beta)}\left(\frac{q_1q_2}{p_1p_2}\right)^i\binom{\frac{N-\alpha+\beta}{2}}{i}\binom{\frac{N+\alpha-\beta}{2}-1}{i+\alpha-1},$$

where L is the set of pairs $\{(0, 0), (1, -1), (1, 1), (2, 0)\}$

and

$$\psi_{\alpha,\beta} = \begin{cases} 1 & \text{for } (\alpha, \beta) = (0, 0) \\ q_2 & \text{for } (\alpha, \beta) = (1, -1) \\ q_1 & \text{for } (\alpha, \beta) = (1, 1) \\ q_1q_2 & \text{for } (\alpha, \beta) = (2, 0) \end{cases} \tag{2.2.36}$$

and where $H(\alpha, \beta)$ and $H'(\alpha, \beta)$ have already been defined after (2.2.12) and (2.2.20).

2.3 Exact Comparison of Procedure R_N and $\bar{R}_n$.

In this section, we introduce a well known procedure [12] quite similar to R_N and denote it by $\bar{R}_n$. We show that the probabilities of correct selection are identical for procedures R_N and $\bar{R}_n$ when the total number N of observations is even.

Procedure $\bar{R}_n$ as opposed to R_N employs the "vector-at-a-time" (VT) sampling rule rather than the PW sampling rule, i.e., with $\bar{R}_n$, we observe a fixed number, n, of vector observations (each vector consisting of one observation on both treatments 1 and 2). At termination, we decide (as in R_N) that the better treatment is the one having the most successes (with the usual randomization, if necessary).

For any event D, let $P_N\{D\}$ be the probability of the event based on a total of N observations $(N \geq 1)$. It is easy to see that for procedure $\bar{R}_n$

$$P_N\{CS\} = P_N\{S_1 - S_2 > 0\} + \frac{1}{2} P_N\{S_1 - S_2 = 0\} \tag{2.3.1}$$

$$= \sum_{j=1}^{N/2} \sum_{i=0}^{N/2-j} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i+j} p_2^{\frac{N}{2}-i-j} p_1^{\frac{N}{2}-i} q_2^{i+j} q_1^{i}$$

$$+ \frac{1}{2} \sum_{i=0}^{N/2} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i} (p_1 p_2)^{\frac{N}{2}-i} (q_1 q_2)^{i},$$

where $N = 2n = 2, 4, 6, \ldots$, and where S_1 (resp., S_2) is the current number of successes with treatment 1 (resp., 2). Let us note that S_1 (resp., S_2) evaluated after N total observations is a binomial random variable with sample size $\frac{N}{2}$ and success parameter p_1 (resp., p_2).

We prove that for $N = 2, 4, 6, 8, \ldots,$

$$P_N\{CS|\overline{R}_n\} = P_N\{CS|R_N\} \tag{2.3.2}$$

by first noting from (2.2.27) and (2.3.1) that

$$P_2\{CS|\overline{R}_n\} = P_2\{CS|R_N\} \tag{2.3.3}$$

and then proving, in most of the remainder of this section, that

$$P_{N+2}\{CS|\overline{R}_n\} - P_N\{CS|\overline{R}_n\} = P_{N+2}\{CS|R_N\} - P_N\{CS|R_N\} \tag{2.3.4}$$

for $N = 2, 4, 6, \ldots$.

Let A_i (resp., B_i) denote the event that treatment 1 (resp., 2) is used on the i^{th} observation. Also, for any two events E and F, let $E \cap F$ denote their intersection. It is then easy to see that for procedure R_N

$$(2.3.5)\qquad P_{N+1}\{CS \cap A_{N+1}\} - P_N\{CS \cap A_{N+1}\}$$

$$= \frac{p_1}{2}\,[P_N\{[S_1 - S_2 = 0] \cap A_{N+1}\} + P_N\{[S_1 - S_2 = -1] \cap A_{N+1}\}].$$

Similarly

$$(2.3.6)\qquad P_{N+1}\{CS \cap B_{N+1}\} - P_N\{CS \cap B_{N+1}\} = [P_{N+1}\{B_{N+1}\} - P_{N+1}\{IS \cap B_{N+1}\}]$$

$$- [P_N\{B_{N+1}\} - P_N\{IS \cap B_{N+1}\}] = -[P_{N+1}\{IS \cap B_{N+1}\}$$

$$- P_N\{IS \cap B_{N+1}\}] = - h_N(p_2, p_1),$$

where the event IS is the complement of the event CS and where $h_N(p_1, p_2)$ is the function of p_1 and p_2 defined by the first line of (2.3.5). Let us note that the event B_{N+1} depends only on the first N observations, which accounts for the second equality of (2.3.6). It thus follows from (2.3.5) and (2.3.6) that for procedure R_N

$$(2.3.7)\qquad P_{N+1}\{CS\} - P_N\{CS\} = h_N(p_1, p_2) - h_N(p_2, p_1)$$

and that

$$(2.3.8)\qquad P_{N+2}\{CS\} - P_N\{CS\} = [P_{N+2}\{CS\} - P_{N+1}\{CS\}] + [P_{N+1}\{CS\} - P_N\{CS\}]$$

$$= [h_{N+1}(p_1, p_2) - h_{N+1}(p_2, p_1)] + [h_N(p_1, p_2) - h_N(p_2, p_1)]$$

$$= [h_{N+1}(p_1, p_2) + h_N(p_1, p_2)] - [h_{N+1}(p_2, p_1) + h_N(p_2, p_1)].$$

We now find explicit expressions, first for $P_N\{[S_1 - S_2 = 0] \cap A_{N+1}\}$ and then for $P_N\{[S_1 - S_2 = -1] \cap A_{N+1}\}$. Based on N observations, the event $A_1 \cap A_{N+1}$ implies that treatments 1 and 2 have an equal number of failures and the event $B_1 \cap A_{N+1}$ implies that treatment 2 has one more failure than treatment 1. Hence in order for the event $[S_1 - S_2 = 0] \cap A_{N+1}$ to occur, we must have that A_1 occurs if N is even and that B_1 occurs if N is odd. Thus if N is even (resp., N is odd), we obtain $P_N\{[S_1 - S_2 = 0] \cap A_{N+1}\}$ by summing over those terms in the expansion of $L_{N,0}$ (resp., $T_{N,0}$) given by (2.2.33), which have p_1 and p_2 raised to equal exponents. Thus by (2.2.12), (2.2.14), (2.2.20) and (2.2.22), for N even,

$$(2.3.9) \qquad P_N\{[S_1 - S_2 = 0] \cap A_{N+1}\} = \frac{1}{2} D_{N,0}^{(0,0)} + \frac{q_1}{2} A_{N,0}^{(1,0)} = \frac{1}{2}(p_1p_2)^{\frac{N}{2}} \sum_{i=1}^{N/2} \left(\frac{q_1q_2}{p_1p_2}\right)^i \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}-1}{i-1}$$

and for N odd

$$(2.3.10) \qquad P_N\{[S_1 - S_2 = 0] \cap A_{N+1}\} = \frac{q_2}{2} A_{N,0}^{(1,0)} + \frac{q_1q_2}{2} A_{N,0}^{(2,0)} = \frac{q_2}{2}(p_1p_2)^{\frac{N-1}{2}} \sum_{i=0}^{N/2-\frac{1}{2}} \left(\frac{q_1q_2}{p_1p_2}\right)^i \binom{\frac{N}{2}-\frac{1}{2}}{i}\binom{\frac{N}{2}-\frac{1}{2}}{i}.$$

Next, we find $P_N\{[S_1 - S_2 = -1] \cap A_{N+1}\}$ by first finding $P_N\{[S_1 - S_2 = 1] \cap B_{N+1}\}$ and then interchanging p_1 and p_2. In order for the event $[S_1 - S_2 = 1] \cap B_{N+1}$ to occur, we must have that A_1 occurs if N is even and that B_1 occurs if N is odd. Thus if

N is even (resp., N is odd), we obtain $P_N\{[S_1 - S_2 = 1] \cap B_{N+1}\}$ by summing over those terms in the expansion of $L_{N,0}$ (resp., $T_{N,0}$) given by (2.2.33), which have p_1 raised to an exponent, one degree higher than p_2 and then multiply by $\frac{1}{2}$. This is equivalent to summing over those terms of $B_{N,0}^{(0,0)} + q_1 C_{N,0}^{(1,0)}$ (resp., $q_2 C_{N,0}^{(1,0)} + q_1 q_2 C_{N,0}^{(2,0)}$) which have p_1 raised to an exponent, one degree higher than p_2 and then multiplying by $\frac{1}{2}$. Thus by (2.2.12), (2.2.14), (2.2.20) and (2.2.22), for N even

$$(2.3.11) \qquad P_N\{[S_1 - S_2 = 1] \cap B_{N+1}\} = \frac{q_1}{2} p_1^{N-1} \sum_{i=0}^{N/2 - 1} \left(\frac{q_1 q_2}{p_1 p_2}\right)^i \binom{\frac{N}{2}}{i} \binom{\frac{N}{2} - 1}{i} \left(\frac{p_2}{p_1}\right)^{\frac{N}{2}-1}$$

and for N odd,

$$(2.3.12) \qquad P_N\{[S_1 - S_2 = 1] \cap B_{N+1}\}$$
$$= \frac{q_1 q_2}{2} p_1^{N-2} \sum_{i=0}^{(N-3)/2} \left(\frac{q_1 q_2}{p_1 p_2}\right)^i \binom{\frac{N}{2} - \frac{1}{2}}{i} \binom{\frac{N}{2} - \frac{1}{2}}{i+1} \left(\frac{p_2}{p_1}\right)^{\frac{N-3}{2}} ;$$

$P_N\{[S_1 - S_2 = -1] \cap A_{N+1}\}$ is then obtained from the rhs of (2.3.11) and (2.3.12) by interchanging p_1 and p_2. We now assume that N is even ($N = 2, 4, 6, 8, \ldots$) and write $t(p_1, p_2) = h_{N+1}(p_1, p_2) + h_N(p_1, p_2)$ explicitly (from (2.3.9), (2.3.10), (2.3.11) and (2.3.12)) as

$$(2.3.13) \qquad t(p_1, p_2) = \frac{p_1 q_2}{4} \Bigg[\sum_{i=1}^{N/2} \binom{\frac{N}{2}}{i} \binom{\frac{N}{2} - 1}{i - 1} (p_1 p_2)^{\frac{N}{2} - i} q_1^i q_2^{i-1}$$
$$+ \sum_{i=0}^{N/2 - 1} \binom{\frac{N}{2}}{i} \binom{\frac{N}{2} - 1}{i} p_1^{N/2 - i - 1} p_2^{N/2 - i} (q_1 q_2)^i$$
$$+ \sum_{i=0}^{N/2} \binom{\frac{N}{2}}{i} \binom{\frac{N}{2}}{i} (p_1 p_2)^{\frac{N}{2} - i} (q_1 q_2)^i$$
$$+ \sum_{i=0}^{N/2 - 1} \binom{\frac{N}{2}}{i} \binom{\frac{N}{2}}{i+1} p_1^{N/2 - i - 1} p_2^{N/2 - i} q_1^{i+1} q_2^i \Bigg].$$

We return and make use of (2.3.8) and (2.3.13) after first finding $P_{N+2}\{CS\} - P_N\{CS\}$ for procedure $\bar{R}_n$.

It is easy to see that for procedure $\bar{R}_n$

$$(2.3.14)\qquad P_{N+2}\{CS\} - P_N\{CS\} = \frac{p_1 q_2}{2}\,[P_N\{S_1 - S_2 = 0\} + P_N\{S_1 - S_2 = -1\}]$$
$$- \frac{p_2 q_1}{2}\,[P_N\{S_1 - S_2 = 0\} + P_N\{S_1 - S_2 = 1\}]$$
$$= m(p_1, p_2) - m(p_2, p_1),$$

where

$$(2.3.15)\qquad m(p_1, p_2) = \frac{p_1 q_2}{2}\,[\sum_{i=0}^{N/2}\binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i}(p_1 p_2)^{\frac{N}{2} - i}(q_1 q_2)^i$$
$$+ \sum_{i=0}^{N/2 - 1}\binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i+1} p_1^{N/2 - i - 1}\, p_2^{N/2 - i}\, q_1^{i+1} q_2^{i}].$$

It is to be noted that N is even, since $N = 2n$, where n is the number of vector observations.

From (2.3.8) and (2.3.14), our proof is completed by proving that $s(p_1, p_2) = t(p_1, p_2) - m(p_1, p_2)$ is a symmetric function of p_1 and p_2, i.e., $s(p_1, p_2) = s(p_2, p_1)$. We proceed to prove this, $s(p_1, p_2)$ can be written as (see (2.3.13) and (2.3.15))

$$(2.3.16)\qquad s(p_1, p_2) = \frac{1}{4}(u_1 + u_2 - u_3 - u_4),$$

where

$$(2.3.17)\qquad u_1 = \sum_{i=1}^{N/2}\binom{\frac{N}{2}}{i}\binom{\frac{N}{2} - 1}{i - 1} p_1^{N/2 - i + 1}\, p_2^{N/2 - i}(q_1 q_2)^i,$$
$$u_2 = \sum_{i=0}^{N/2 - 1}\binom{\frac{N}{2}}{i}\binom{\frac{N}{2} - 1}{i}(p_1 p_2)^{\frac{N}{2} - i} q_1^{i} q_2^{i+1},$$

$$u_3 = \sum_{i=0}^{N/2} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i} p_1^{N/2\ -i+1} p_2^{N/2\ -i} q_1^{i} q_2^{i+1},$$

$$u_4 = \sum_{i=0}^{N/2\ -1} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}}{i+1} (p_1p_2)^{\frac{N}{2}\ -i} (q_1q_2)^{i+1}.$$

$u_1 + u_2 - u_3$ is a symmetric function of p_1 and p_2 since

$$(2.3.18)\quad u_1 + u_2 - u_3 = u_1 + \sum_{i=1}^{N/2\ -1} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}-1}{i} (p_1p_2)^{\frac{N}{2}\ -i} q_1^{i} q_2^{i+1}[1-p_1]$$

$$- \sum_{i=1}^{N/2\ -1} \binom{\frac{N}{2}}{i}\left[\binom{\frac{N}{2}}{i} - \binom{\frac{N}{2}-1}{i}\right] p_1^{N/2\ -i+1} p_2^{N/2\ -i} q_1^{i} q_2^{i+1} + (p_1p_2)^{\frac{N}{2}} q_2[1-p_1] - p_1 q_1^{\frac{N}{2}} q_2^{\frac{N}{2}+1}$$

$$= \sum_{i=1}^{N/2\ -1} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}-1}{i} (p_1p_2)^{\frac{N}{2}\ -i} (q_1q_2)^{i+1} + (p_1p_2)^{\frac{N}{2}} q_1q_2$$

$$+ \sum_{i=1}^{N/2\ -1} \binom{\frac{N}{2}}{i}\binom{\frac{N}{2}-1}{i-1} p_1^{N/2\ -i+1} p_2^{N/2\ -i} (q_1q_2)^{i}[1-q_2] + p_1(q_1q_2)^{\frac{N}{2}}[1-q_2].$$

For the second equality of (2.3.18), we make use of the identity

$$(2.3.19)\quad \binom{\frac{N}{2}}{i} - \binom{\frac{N}{2}-1}{i} = \binom{\frac{N}{2}-1}{i-1}.$$

Thus, the fact that u_4 is symmetric in (p_1, p_2) implies that $s(p_1, p_2) = s(p_2, p_1)$ and our proof is now completed.

Let N_{PW} and N_{VT} be the smallest value of N necessary to insure (1.1.1) for the PW and VT sampling rules. From (2.3.2), we know that

$$(2.3.20)\quad 0 \le N_{VT} - N_{PW} \le 1;$$

the reason that $N_{VT} - N_{PW}$ is not identically equal to zero follows from the fact that with procedure $\bar{R}_n$, the total number of observations must always be even. It thus follows that we have no special preference for either procedure R_N or $\bar{R}_n$, when the total number of observations is used as our criterion. Suppose, though our interest is in minimizing the expected number of observations with the poorer treatment, EN_2, rather than the total number of observations. Let $EN_{2,PW}$ (resp., $EN_{2,VT}$) be the expected number of observations with the poorer treatment for the PW (resp., VT) sampling rule, when there are N_{PW} (resp., N_{VT}) total observations. It is obviously true that

$$(2.3.21) \qquad EN_{2,VT} = N_{2,VT} = \frac{N_{VT}}{2} .$$

It is also easy to see that for large N, the expected number of observations with the poorer treatment (using the PW sampling rule) is EZ_{q_2}/EZ_{q_1} times the number of observations with the better treatment $(= \frac{q_1}{q_2} EN_{1,PW})$, where the random variable Z_q is a geometric rv representing the number of trials until (and including) the first failure, where the probability of a failure on a single trial is equal to q. This yields the result that

$$(2.3.22) \qquad EN_{2,PW} \approx \frac{q_1}{q_1 + q_2} N_{PW} .$$

Using the result (2.3.20), we have (for large N) by (2.3.21) and (2.3.22) that

$$(2.3.23) \qquad \frac{EN_{2,PW}}{EN_{2,VT}} \approx \frac{2q_1}{q_1 + q_2} ,$$

which holds for any configuration of the true values (q_1, q_2). Hence, asymptotically, $EN_{2,PW}$ is uniformly smaller than $EN_{2,VT}$ for all pairs (q_1, q_2) with $q_2 > q_1$. With regards to the minimization of EN_2, PW-sampling would therefore be preferable to VT-sampling for the fixed sample size selection problem with 2 populations.

2.4 Asymptotic Results for Procedure $\bar{R}_n$ (or R_N).

We find, for large n, an approximation of the smallest value of n insuring (1.1.1) for procedure $\bar{R}_n$. Let us call this approximate value, n_s; the corresponding value of N is then equal to $2n_s$. By (2.3.20), we have at the same time determined that for large N, the smallest value of N necessary to satisfy (1.1.1) for procedure R_N is approximately $2n_s$. However, an independent derivation of this asymptotic value of N for procedure R_N is then made; it illustrates a technique which proves very useful in Section 2.6.

Let $B(n, p)$ represent a binomial random variable with sample size n and success parameter p and let U_n be a random variable (rv) with distribution

$$U_n \sim B(n, p_1) - B(n, p_2). \tag{2.4.1}$$

We note that the expected value (EU_n) and the variance $(\text{Var } U_n)$ are given by

$$EU_n = n(p_1 - p_2), \text{ Var } U_n = n[p_1(1-p_1) + p_2(1-p_2)]. \tag{2.4.2}$$

Also, let E_0 (resp., E_1) denote the event that at termination the number of successes achieved by treatment 1 is equal to (resp., strictly greater than) the number of successes achieved by treatment 2. For large n it is seen by the central limit theorem that for pairs (p_1, p_2) such that $p_1 - p_2 \geq \Delta^*$,

$$(2.4.3) \quad P\{CS|\bar{R}_n\} = P\{E_1\} + \frac{1}{2} P\{E_0\}$$

$$= P\{B(n, p_1) > B(n, p_2)\} + \frac{1}{2} P\{B(n, p_1) = B(n, p_2)\}$$

$$= P\{W_n > -\frac{\sqrt{n}\,(p_1-p_2)}{\sqrt{p_1(1-p_1)+p_2(1-p_2)}}\} \quad -$$

$$+ \frac{1}{2} P\{W_n = \frac{-\sqrt{n}\,(p_1-p_2)}{\sqrt{p_1(1-p_1)+p_2(1-p_2)}}\}$$

$$\geq P\{W_n > -A_1\} + \frac{1}{2} P\{W_n = -A_1\}$$

$$\sim \Phi(A_1) ,$$

where

$$(2.4.4) \quad A_1 = \frac{\sqrt{n}\,\Delta^*}{\sqrt{p_1(1-p_1)+p_2(1-p_2)}}$$

with $p_1 - p_2 = \Delta^*$, where

$$(2.4.5) \quad W_n = \frac{U_n - EU_n}{\sqrt{\mathrm{Var}\ U_n}} ,$$

and where $\Phi(x)$ is the standard normal cumulative distribution function (cdf). In the fourth expression of (2.4.3), we have set $p_1 - p_2 = \Delta^*$, since for any fixed value of p_2 the probability of correct selection is made smaller by letting p_1 approach p_2. Also, it is assumed that as $n \to \infty$, $\Delta^* \to 0$ at a speed so that $\Delta^* \sqrt{n}$ remains bounded and, thus, the last line of (2.4.3) is a valid application of the central limit theorem.

We find the pair (p_1, p_2) with $p_1 - p_2 \geq \Delta^*$ which minimizes $P\{CS|\bar{R}_n\}$ for large n. We call this pair the "least favorable" (LF) configuration. By (2.4.3), we note that the problem of minimizing the $P\{CS|\bar{R}_n\}$ for large n is the same as minimizing A_1 or maximizing

$$(2.4.6) \qquad T_1 = \frac{n(\Delta^*)^2}{A_1^2} = p_1(1-p_1) + p_2(1-p_2)$$

with $p_1 - p_2 = \Delta^*$ and this is equivalent to maximizing

$$(2.4.7) \qquad T_1(p_0) = 2p_0(1-p_0) - \frac{(\Delta^*)^2}{2} ,$$

where $\frac{\Delta^*}{2} \leq p_0 \leq 1 - \frac{\Delta^*}{2}$ is the center point of the interval (p_2, p_1), i.e., $p_1 = p_0 + \frac{\Delta^*}{2}$ and $p_2 = p_0 - \frac{\Delta^*}{2}$.

Writing $T_1(p_0)$ only up to order $\mathcal{O}(\Delta^{*2})$ accuracy (since Δ^* is assumed to be small) as

$$(2.4.8) \qquad T_1(p_0) = 2p_0(1-p_0) + \mathcal{O}(\Delta^{*2}),$$

we find that $T_1(p_0)$ is maximized at $p_0 = \frac{1}{2}$, having maximum value

$$(2.4.9) \qquad T_1(\tfrac{1}{2}) \approx \frac{1}{2} .$$

Thus for large n, small Δ^* and pairs (p_1, p_2) such that $p_1 - p_2 \geq \Delta^*$,

$$(2.4.10) \qquad P\{CS|\bar{R}_n\} \geq P_{LF}\{CS|\bar{R}_n\} \sim \min_{p_0} \Phi(\Delta^*\sqrt{\frac{n}{T_1(p_0)}}) = \Phi(\Delta^*\sqrt{2n}),$$

where $P_{LF}\{CS\}$ is the probability of correct selection under the LF configuration. Hence we find, for large n and small Δ^*, the smallest value of n satisfying (1.1.1) for procedure $\bar{R}_n$ by setting $\Phi(\Delta^*\sqrt{2n}) = P^*$, i.e.,

$$(2.4.11) \qquad n = \frac{\lambda^2}{2(\Delta^*)^2} ,$$

where $\lambda = \lambda(P^*)$ is such that $\Phi(\lambda) = P^*$.

We now obtain an asymptotic formulation for the probability of correct selection under procedure R_N independent of the result in (2.4.10). For large values of N, procedures R_N and R'_N are approximately the same; since they possibly differ only when the absolute difference between the successes of treatments 1 and 2 at termination is less than or equal to one. Therefore, asymptotically, it is necessary to work with only one of them.

Define a <u>complete turn of 1</u> to be each unbroken sequence of observations on treatment 1 which consists of a number of successes followed by a failure. Define a <u>complete turn of 2</u> in a similar manner. Let $K(1, N)$ be the number of complete turns of 1 when we stop our experiment after the N^{th} observation. Define $K(2,N)$ in a similar manner.

We shall prove that

$$\text{(2.4.12)} \qquad \frac{K(2, N)}{N} \xrightarrow{a.s.} \left(\frac{1}{q_1} + \frac{1}{q_2}\right)^{-1},$$

where $M_N \xrightarrow{a.s.} c$ means that the sequence of random variables M_N converges almost surely to the constant c.

We prove that if at the N^{th} observation, we are in a situation where treatment 1 is used on the first and last observation, then for N sufficiently large, we can show that $\frac{K(2, N)}{N}$ is arbitrarily close to $\left(\frac{1}{q_1} + \frac{1}{q_2}\right)^{-1}$; we can similarly prove the above result for other possible situations, e.g., treatment 2 is used on the first and last observation. This will prove (2.4.12).

Let Z_q be a geometric random variable representing the number of trials until (and including) the first failure, where the probability of a failure on a single trial is equal to q. Z_q has the probability law, $P\{Z_q = z\} = q(1-q)^{z-1}$ for $z = 1, 2, \ldots,$ with mean and variance given by, $EZ_q = \frac{1}{q}$ and $\operatorname{Var} Z_q = \frac{p}{q^2}$ $(p = 1-q)$. Let X_i (resp., Y_i) be equal to the number of observations with with treatment 1 (resp., 2) in its i^{th} complete turn. We note that the X_i(resp., Y_i) are independent and identically distributed random variables with the same distribution as Z_{q_1} (resp., Z_{q_2}). If treatment 1 is used on the first and last observation, then

$$(2.4.13) \quad X_1 + X_2 + \ldots + X_{K(2,\, N)} + Y_1 + Y_2 + \ldots + Y_{K(2,\, N)} = N - C_N,$$

where C_N is a non-negative random variable such that

$$(2.4.14) \quad P\{C_N \leq x\} \geq P\{Z_{q_1} \leq x\}$$

for all real x. (For the sake of convenience, we can define $C_N = 0$ for all those N such that A is not used on the N^{th} observation.)

It follows from (2.4.14) that C_N has a finite expectation, since Z_{q_1} has a finite expectation. Hence it is easily shown that

$$(2.4.15) \quad \frac{C_N}{N} \xrightarrow{a.s.} 0 \; ;$$

the proof, which is based on the Borel-Cantelli Lemma (see Loeve [7]), is omitted.

From (2.4.13), we have that

$$(2.4.16) \quad \frac{K(2,\, N)}{N} \left(\frac{X_1 + X_2 + \ldots + X_{K(2,N)}}{K(2,N)} + \frac{Y_1 + Y_2 + \ldots + Y_{K(2,N)}}{K(2,N)}\right) = 1 - \frac{C_N}{N} \; .$$

By the Strong Law of Large Numbers (SLLN),

$$(2.4.17)\qquad \frac{X_1+X_2+\ldots+X_{K(2,N)}}{K(2,N)} \xrightarrow{a.s.} \frac{1}{q_1}\;, \quad \frac{Y_1+Y_2+\ldots+Y_{K(2,N)}}{K(2,N)} \xrightarrow{a.s.} \frac{1}{q_2}\;,$$

since

$$(2.4.18)\qquad K(2,N) \xrightarrow{a.s.} \infty \quad \text{as} \quad N \to \infty.$$

Therefore, by (2.4.15) and (2.4.16), we have proven (2.4.12) for the case in which treatment 1 is used on the first and last observation. The remaining proofs are similar and hence this proves (2.4.12).

Since the absolute difference between $K(1, N)$ and $K(2, N)$ is at most one, we have (for large N) that

$$(2.4.19)\qquad K(1, N) \approx K(2, N) \approx \frac{N}{\frac{1}{q_1}+\frac{1}{q_2}}\;.$$

Hence, for large N,

$$(2.4.20)\qquad P\{CS|R_N\} \sim P\{\sum_{i=1}^{K(2,N)} (X_i - Y_i) > 0\}$$

$$= P\{W_N > -(q_2 - q_1)\sqrt{\frac{K(2,N)}{p_1q_2^2 + p_2q_1^2}}\}$$

$$\sim P\{W_N > -(q_2 - q_1)\sqrt{\frac{q_1q_2N}{(q_1+q_2)(p_1q_2^2+p_2q_1^2)}}\},$$

where

$$(2.4.21)\qquad W_N = \frac{\sum_{i=1}^{K(2,N)} [(X_i - Y_i) - (\frac{1}{q_1} - \frac{1}{q_2})]}{\sqrt{K(2,N)[\frac{p_1}{q_1^2} + \frac{p_2}{q_2^2}]}}\;.$$

At this point, a theorem proved in a paper by Renyi (see [8]) is useful. It can be stated as follows:

Theorem R:

Let us suppose that $h_1, h_2, \ldots, h_n, \ldots$ are independent and identically distributed random variables with mean value 0 and variance 1. Let us put $H_n = h_1 + h_2 + \ldots + h_n$. Let further $a(t)$ denote a positive integer random variable for any $t > 0$ such that $\frac{a(t)}{t}$ converges for $t \to +\infty$ in probability to a constant $c > 0$. Then we have

$$(2.4.22) \qquad \lim_{t \to +\infty} P\{\frac{H_{a(t)}}{\sqrt{a(t)}} < x\} = \Phi(x).$$

Let us note that, as usual, $\Phi(x)$ is the standard normal cdf. Thus, from(2.4.20),we have for any pair (q_1, q_2) with $q_2 - q_1 \geq \Delta^*$ and for $N \to \infty$ that

$$(2.4.23) \qquad \begin{aligned} P\{CS|R_N\} &\geq P\{W_N > -A_2\} \\ &\sim \Phi(A_2), \end{aligned}$$

where

$$(2.4.24) \qquad A_2 = \Delta^* \sqrt{\frac{q_1 q_2 N}{(q_1+q_2)(p_1 q_2^2 + p_2 q_1^2)}} ,$$

with $q_2 - q_1 = \Delta^*$. The first line of (2.4.23) follows from the fact that for any fixed value of q_1, the probability of correct selection is made smaller by letting q_2 approach q_1. In order to apply Theorem R in the second line of (2.4.23), we assume that as $N \to \infty$, $\Delta^* \to 0$ in a manner so that $\Delta^* \sqrt{N}$ remains bounded. The problem of minimizing the $P\{CS|R_N\}$ for large N, over pairs

(q_1, q_2) such that $q_2 - q_1 \geq \Delta^*$ is thus reduced to maximizing

$$(2.4.25) \quad T_2 = \frac{N(\Delta^*)^2}{A_2^2} = \frac{(q_2 + q_1)(p_1 q_2^2 + p_2 q_1^2)}{q_1 q_2}$$

with $q_2 - q_1 = \Delta^*$ and this is equivalent to maximizing

$$(2.4.26) \quad T_2(q_0) = \frac{-4q_0^4 + 4q_0^3 + (\Delta^*)^2 q_0^2 + (\Delta^*)^2 q_0}{q_0^2 - \frac{(\Delta^*)^2}{4}},$$

where $\frac{\Delta^*}{2} \leq q_0 \leq 1 - \frac{\Delta^*}{2}$ is the center point of the interval (q_1, q_2), i.e., $q_2 = q_0 + \frac{\Delta^*}{2}$ and $q_1 = q_0 - \frac{\Delta^*}{2}$.

Taking the derivative of $T_2(q_0)$ with respect to (wrt) q_0, we find that

$$(2.4.27) \quad \frac{dT_2(q_0)}{dq_0} = \frac{-8q_0^5 + 4q_0^4 + 4(\Delta^*)^2 q_0^3 - 4(\Delta^*)^2 q_0^2 - \frac{(\Delta^*)^4}{2} q_0 - \frac{(\Delta^*)^4}{4}}{(q_0^2 - \frac{(\Delta^*)^2}{4})^2}.$$

Writing $T_2(q_0)$ and $\frac{dT_2(q_0)}{dq_0}$ only up to order $\Theta(\Delta^{*2})$ accuracy (since Δ^* is assumed to be small) as

$$(2.4.28) \quad T_2(q_0) = -4q_0^2 + 4q_0 + \Theta(\Delta^{*2}),$$

$$\frac{dT_2(q_0)}{dq_0} = -8q_0 + 4 + \Theta(\Delta^{*2}),$$

we find that, for small Δ^*, $T_2(q_0)$ is maximized at $q_0 = \frac{1}{2}$ and the maximum value of $T_2(q_0)$ is approximately

$$(2.4.29) \quad T_2(\tfrac{1}{2}) \approx 1.$$

We thus have for large N and for pairs (p_1, p_2) with $(p_1 - p_2) \geq \Delta^*$ (correct up to order $\Theta(\Delta^{*2})$) that

$$(2.4.30)\qquad P\{CS|R_N\} \geq P_{LF}\{CS|R_N\} \sim \min_{q_0} \Phi\left(\Delta^* \sqrt{\frac{N}{T_2(q_0)}}\right) = \Phi(\Delta^* \sqrt{N}).$$

Hence for large N (and small Δ^*), we find the smallest value of N satisfying (1.1.1) for procedure R_N by setting $\Phi(\Delta^* \sqrt{N}) = P^*$, i.e.,

$$(2.4.31)\qquad N = \left(\frac{\lambda}{\Delta^*}\right)^2,$$

where $\lambda = \lambda(P^*)$ is defined after (2.4.11).

2.5 Three-Decision Problem.

For the remainder of this chapter, we consider the "three-decision problem," i.e., at the termination of sampling, we have three decisions:

d_1) that A is better,

d_2) that A and B are equally good,

d_3) that B is better.

In the present context, we consider (for large N) both the PW and VT sampling rules under our fixed sample size termination rule and then compare them.

2.6 Asymptotic Results under PW Sampling with Fixed Sample Size N.

Suppose that we employ the PW sampling rule with a fixed total number of observations. Let S_{AN} (resp., S_{BN}) be the number of successes with treatment A (resp., B) up to and including the N^{th} observation. Let M_N be equal to $S_{AN} - S_{BN}$. We decide on the relative merits of treatments A and B in the following manner:

(i) If $\frac{M_N}{N} > f$, then we make decision d_1,

(ii) If $-f \leq \frac{M_N}{N} \leq f$, then we make decision d_2,

(iii) If $\frac{M_N}{N} < -f$, then we make decision d_3,

where f is a non-negative constant. Let $M_N^* = S_{1N} - S_{2N}$.

Let D_1 denote the event that $\frac{M_N^*}{N} > f$, so that a correct selection is made when this event occurs; let D_2 denote the event that $-f \leq \frac{M_N^*}{N} \leq f$, so that we decide that the two treatments are equally good. We choose N and f such that the following two conditions are satisfied:

$$(2.6.1) \qquad \text{(i)} \quad P\{D_1\} \geq P_1^* \quad \text{if} \quad p_1 - p_2 \geq \Delta^*,$$

$$\text{(ii)} \quad P\{D_2\} \geq P_2^* \quad \text{if} \quad p_1 = p_2 \leq \theta,$$

where P_1^*, P_2^*, Δ^* and θ are positive constants such that

$$(2.6.2) \qquad \tfrac{1}{2} < P_1^* < 1,\ \tfrac{1}{2} < P_2^* < 1,\ 0 < \Delta^* < 1,\ \frac{1+\Delta^*}{2} \leq \theta < 1 \ ;$$

the necessity for introducing θ becomes clearer later in the section.

Letting $K(2, N)$, X_i, Y_i and W_N be as defined in Section 2.4, we find that for large N and for any pair (q_1, q_2) such that $q_2 - q_1 \geq \Delta^*$ (in a manner similar to the derivation of (2.4.20) and (2.4.23)),

$$(2.6.3) \qquad P\{D_1\} = P\{M_N^* > Nf\} \sim P\{\sum_{i=1}^{K(2,N)} (X_i - Y_i) > Nf\}$$

$$= P\{W_N > \frac{q_1q_2[Nf - K(2,N)(q_2-q_1)(q_1q_2)^{-1}]}{\sqrt{K(2,N)[p_1q_2^2+p_2q_1^2]}}\}$$

$$\sim P\{W_N > -\left(\sqrt{\frac{q_1q_2N}{(p_1q_2^2+p_2q_1^2)(q_1+q_2)}}\right)[(q_2-q_1) - f(q_1+q_2)]\}$$

$$\geq P\{W_N > -t_2(q_1; q_2)\}$$

$$\sim \Phi(t_2(q_1; q_2)) \ ,$$

where Φ is the standard normal cdf and where

$$(2.6.4) \qquad t_2(q_1;\, q_2) = \sqrt{\frac{q_1 q_2 N}{(p_1 q_2^2 + p_2 q_1^2)(q_1 + q_2)}}\,[\Delta^* - f(q_1 + q_2)],$$

with $q_2 - q_1 = \Delta^*$. Let us note that as in the previous sections, we assume that as $N \to \infty$, $\Delta^* \to 0$ in a manner so that $\Delta^* \sqrt{N}$ remains bounded.

In order to satisfy (i) of (2.5.1), we now find the $(q_1;\, q_2)$ configuration, say $(q_{1,0};\, q_{2,0})$, which for large N minimizes $P\{D_1\}$ subject to the condition that $q_2 - q_1 \geq \Delta^*$. Let us denote this configuration by "LF(D_1) i.e., it is least favorable wrt the probability requirement on the event D_1. We assert that a pair $(N,\, d)$ will for large N satisfy condition (i) if it satisfies the equation

$$(2.6.5) \qquad \Phi(t_2(q_{1,0};\, q_{2,0})) = P_1^*.$$

As in the previous sections, our problem is reduced to maximizing

$$(2.6.6) \qquad H_2 = \frac{N}{[t_2(q_1;\, q_2)]^2} = \frac{(q_1 + q_2)(p_1 q_2^2 + p_2 q_1^2)}{q_1 q_2 [\Delta^* - f(q_1 + q_2)]^2}$$

with $q_2 - q_1 = \Delta^*$ and this is equivalent to maximizing

$$(2.6.7) \qquad H_2(q_0) = \frac{2q_0\left[-2q_0^3 + 2q_0^2 + \frac{(\Delta^*)^2}{2} q_0 + \frac{(\Delta^*)^2}{2}\right]}{\left[q_0^2 - \frac{(\Delta^*)^2}{4}\right][\Delta^* - 2fq_0]^2},$$

where $\frac{\Delta^*}{2} \leq q_0 \leq 1 - \frac{\Delta^*}{2}$ and q_0 is the center point of the interval (q_1, q_2), i.e., $q_2 = q_0 + \frac{\Delta^*}{2}$ and $q_1 = q_0 - \frac{\Delta^*}{2}$. We note that in order for condition (i) to be achieved, we must have $\Delta^* - 2fq_0 > 0$ for all

q_0 between $\frac{\Delta^*}{2}$ and $1 - \frac{\Delta^*}{2}$, i.e., $f < \frac{\Delta^*}{2 - \Delta^*}$; f is therefore restricted to lie in the half-open interval $[0, \frac{\Delta^*}{2 - \Delta^*})$.

Let us define $T_2(q_0)$ by

$$(2.6.8) \qquad T_2(q_0) = H_2(q_0)(\Delta^*)^2 = \frac{2q_0[-2q_0^3 + 2q_0^2 + \frac{(\Delta^*)^2}{2} q_0 + \frac{(\Delta^*)^2}{2}]}{[q_0^2 - \frac{(\Delta^*)^2}{4}][1 - cq_0]^2}$$

where $c = \frac{2f}{\Delta^*}$. Writing $T_2(q_0)$ only up to order $\mathcal{O}(\Delta^{*2})$ accuracy (since Δ^* is assumed to be small) as

$$(2.6.9) \qquad T_2(q_0) = \frac{2}{[1 - cq_0]^2} [-2q_0^2 + 2q_0] + \mathcal{O}(\Delta^{*2}),$$

we find that the derivative of $T_2(q_0)$ is approximately $(\Delta^* \to 0)$

$$(2.6.10) \qquad \frac{dT_2(q_0)}{dq_0} \approx \frac{2}{[1 - cq_0]^2} [-4q_0 + 2] + \frac{4c}{[1 - cq_0]^3} [-2q_0^2 + 2q_0].$$

Setting $\frac{dT_2(q_0)}{dq_0} = 0$, we arrive at the equation

$$(2.6.11) \qquad (2-c)q_0 = 1.$$

Since q_0 must be less than $1 - \frac{\Delta^*}{2}$, we find that the maximum value of $T_2(q_0)$ occurs at

$$(2.6.12) \qquad q_0 = \begin{cases} \frac{1}{2 - c} & \text{for } 0 \leq c \leq \frac{2 - 2\Delta^*}{2 - \Delta^*} \\ 1 - \frac{\Delta^*}{2} & \text{for } \frac{2 - 2\Delta^*}{2 - \Delta^*} \leq c < \frac{\Delta^*}{2 - \Delta^*} \frac{2}{\Delta^*} \end{cases},$$

which leads us to a discussion of two cases, depending on whether c is less than or bigger than $\frac{2 - 2\Delta^*}{2 - \Delta^*}$.

<u>Case A</u>.

For Case A we assume that $0 \leq c \leq \frac{2 - 2\Delta^*}{2 - \Delta^*}$. From (2.6.12), the maximum value of $T_2(q_0)$ is for small Δ^* approximately

$$(2.6.13)\qquad T_2\left(\frac{1}{2-c}\right) \approx 2\left[\frac{2-c}{2-2c}\right]^2\left[\frac{-2}{(2-c)^2} + \frac{2}{2-c}\right] = \frac{1}{1-c}\ .$$

We therefore have for $N \to \infty$, $\Delta^* \to 0$ and pairs (p_1, p_2) such that $p_1 - p_2 \geq \Delta^*$,

$$(2.6.14)\qquad P\{D_1\} \geq P_{LF(D_1)}\{D_1\} \sim \min_{q_0} \Phi\left(\Delta^*\sqrt{\frac{N}{T_2(q_0)}}\right)$$

$$= \Phi(\Delta^*\sqrt{N(1-c)}\ ).$$

<u>Case B.</u>

Let us now assume $\frac{2-2\Delta^*}{2-\Delta^*} \leq c < \frac{2}{2-\Delta^*}$. In this case, by (2.6.12), the maximum value of $T_2(q_0)$ [up to terms of order $\mathcal{O}(\Delta^{*2})$] is

$$(2.6.15)\qquad T_2\left(1-\frac{\Delta^*}{2}\right) \approx 2\left[\frac{1}{1-(1-\frac{\Delta^*}{2})c}\right]^2 \left[-2\left(1-\frac{\Delta^*}{2}\right)^2 + 2\left(1-\frac{\Delta^*}{2}\right)\right]$$

$$\approx \frac{2\Delta^*}{[1-(1-\frac{\Delta^*}{2})c]^2}\ .$$

Hence, asymptotically $(N \to \infty)$, for pairs (p_1, p_2) such that $p_1 - p_2 \geq \Delta^*$ with Δ^* small,

$$(2.6.16)\qquad P\{D_1\} \geq P_{LF(D_1)}\{D_1\} \sim \min_{q_0} \Phi\left(\Delta^*\sqrt{\frac{N}{T_2(q_0)}}\right)$$

$$= \Phi\left(\left[1-\left(1-\frac{\Delta^*}{2}\right)c\right]\sqrt{\frac{N\Delta^*}{2}}\ \right).$$

Later in this section, we shall use these results (2.6.14) and (2.6.16) to determine N and f simultaneously.

Let us now look at condition (ii) of (2.6.1). Letting p denote the common value of $p_1 = p_2$ $(q = 1 - p)$, we observe that

$$(2.6.17) \quad P\{D_2 | p_1 = p_2\} = P\{-Nf \leq M_N^* \leq Nf | p_1 = p_2\}$$

$$\sim P\{-Nf \leq \sum_{i=1}^{K(2,N)} (X_i - Y_i) \leq Nf | p_1 = p_2\}$$

$$= P\{ \frac{-Nf}{\sqrt{2K(2,N)\frac{p}{q^2}}} \leq J_N \leq \frac{Nf}{\sqrt{2K(2,N)\frac{p}{q^2}}}\}$$

$$\sim P\{-f\sqrt{N\frac{q}{p}} \leq J_N \leq f\sqrt{N\frac{q}{p}}\}$$

$$\sim 2\Phi(f\sqrt{N\frac{q}{p}}) - 1,$$

where

$$(2.6.18) \quad J_N = \frac{\sum_{i=1}^{K(2,N)} (X_i - Y_i)}{\sqrt{2K(2,N)\frac{p}{q^2}}} .$$

The notation and derivation of (2.6.17) are similar to that of (2.4.20) and (2.4.23).

We now find for large N the minimum of $P\{D_2\}$ over pairs (p_1, p_2) with $p_1 = p_2 = p$. It is noted that condition (ii) cannot be satisfied unless p is restricted and bounded away from 1. It is thus assumed that $p < \theta$, where θ is a positive constant such that

$$(2.6.19) \quad \frac{1 + \Delta^*}{2} \leq \theta < 1.$$

We choose $\frac{1 + \Delta^*}{2}$ as the smallest possible value of θ for the sake of convenience, since at $c = 0$, the minimum value of $t_2(q_1; q_2)$ occurs at $(q_1;q_2)=(\frac{1-\Delta^*}{2};\frac{1+\Delta^*}{2})$. From (2.6.17), minimizing $P\{D_2\}$ is equivalent $(N \to \infty)$ to minimizing $f\sqrt{N\frac{q}{p}}$ and, thus, we take p as large as possible (namely, $p = \theta$) to do so. Hence, asymptotically $(N \to \infty)$, for (p_1, p_2) configurations with $p_1 = p_2 = p \leq \theta$,

$$(2.6.20) \qquad P\{D_2\} \geq P_{LF(D_2)}\{D_2\} \sim \min_p 2\Phi(f\sqrt{N\tfrac{q}{p}}) - 1$$

$$= 2\Phi(f\sqrt{N\psi}) - 1,$$

where $\psi = \frac{1-\theta}{\theta}$ and where "$LF(D_2)$" denotes the least favorable configuration wrt the probability requirement on the event D_2. Thus, if in addition to (2.6.5), the pair (N, f) satisfied the equation

$$(2.6.21) \qquad \Phi(f\sqrt{N\psi}) = \frac{P_2^* + 1}{2},$$

then it would be asymptotically $(N \to \infty)$ optimal in the sense that it would minimize the N-value necessary to satisfy both conditions (i) and (ii) of (2.6.1). In finding this pair (N, f), we are led to a separate discussion for each of the cases 1 and 2 defined above.

Case A.

For Case A we have $0 \leq c \leq \frac{2-2\Delta^*}{2-\Delta^*}$. From (2.6.14) and (2.6.21), we wish to find the pair (N, f) satisfying the equations:

$$(2.6.22) \qquad \Phi(\Delta^*\sqrt{N(1-c)}) = P_1^*,$$
$$\Phi(f\sqrt{N\psi}) = \frac{P_2^* + 1}{2}.$$

We therefore find that the pair (N, f) must satisfy the two equations

$$(2.6.23) \qquad f^2N\psi = \lambda_2^2 \; ; \; (\Delta^*)^2N(1 - \frac{2f}{\Delta^*}) = \lambda_1^2,$$

where λ_1 and λ_2 are the $100P_1^{*\text{th}}$ and $100(\frac{P_2^* + 1}{2})^{\text{th}}$ percentile points of the standard normal cdf. Solving the two equations in (2.6.23), we arrive at the solution:

$$(2.6.24) \qquad f = \frac{\Delta^*\lambda_2[-\lambda_2 + \sqrt{\lambda_2^2 + \lambda_1^2\psi}\,]}{\lambda_1^2\psi},$$

$$N = \frac{\lambda_1^4\psi}{(\Delta^*)^2} \frac{1}{[-\lambda_2 + \sqrt{\lambda_2^2 + \lambda_1^2\psi}\,]^2} = \frac{[\lambda_2 + \sqrt{\lambda_2^2 + \lambda_1^2\psi}\,]^2}{(\Delta^*)^2\psi} .$$

The above solution for f holds if and only if the f-value obtained, f_0 (say), is such that

$$(2.6.25) \qquad f_0 \leq (\frac{2 - 2\Delta^*}{2 - \Delta^*}) \frac{\Delta^*}{2} = \frac{(1-\Delta^*)\Delta^*}{2 - \Delta^*} .$$

The inequality (2.6.25) holds (for small Δ^*) if and only if

$$(2.6.26) \qquad \psi \geq \frac{\Delta^*(2-\Delta^*)}{(1-\Delta^*)^2} \frac{\lambda_2^2}{\lambda_1^2} \approx 2\Delta^* \frac{\lambda_2^2}{\lambda_1^2} .$$

Since θ (see (2.6.1) and (2.6.2)) is bounded below by $\frac{1 + \Delta^*}{2}$, $\psi = \frac{1 - \theta}{\theta}$ is bounded above by $\frac{1 - \Delta^*}{1 + \Delta^*}$. From (2.6.25) and (2.6.26), it thus follows that f_0 is the asymptotically optimal solution for the problem if $2\Delta^*\lambda_2^2/\lambda_1^2 \leq \psi \leq \frac{1 - \Delta^*}{1 + \Delta^*}$ or equivalently if $\frac{1 + \Delta^*}{2} \leq \theta = \frac{1}{1 + \psi} \leq \lambda_1^2/(\lambda_1^2 + 2\Delta^*\lambda_2^2)$.

Case B.

Let us now consider the case $\frac{2 - 2\Delta^*}{2 - \Delta^*} \leq c < \frac{2}{2 - \Delta^*}$. From (2.6.16) and (2.6.21), the equations we wish to solve for N and f are

$$(2.6.27) \qquad \Phi\left([1-(1-\tfrac{\Delta^*}{2})c]\sqrt{\tfrac{\Delta^* N}{2}}\right) = P_1^* ,$$

$$\Phi(f\sqrt{N\psi}) = \frac{P_2^* + 1}{2} ,$$

and the solutions are easily seen to be

$$(2.6.28) \qquad f = \frac{\Delta^*\lambda_2}{\lambda_2(2-\Delta^*) + \lambda_1\sqrt{2\psi\Delta^*}} ,$$

$$N = \frac{[\lambda_2(2-\Delta^*) + \lambda_1\sqrt{2\psi\Delta^*}\,]^2}{\psi(\Delta^*)^2} .$$

The results in (2.6.28) hold (since c must be greater than $\frac{2-2\Delta^*}{2-\Delta^*}$ for case B) if and only if

$$(2.6.29) \qquad \frac{2\lambda_2}{\lambda_2(2-\Delta^*) + \lambda_1\sqrt{2\psi\Delta^*}} \geq \frac{2-2\Delta^*}{2-\Delta^*} .$$

This inequality (2.6.29) takes the asymptotic form $(\Delta^* \to 0)$

$$(2.6.30) \qquad \psi \leq \frac{1}{2}\,\frac{\lambda_2^2}{\lambda_1^2}\,\frac{\Delta^*(2-\Delta^*)^2}{(1-\Delta^*)^2} \approx 2\Delta^*\,\frac{\lambda_2^2}{\lambda_1^2} .$$

Since θ is bounded above away from 1, ψ is bounded below away from zero. From (2.6.29) and (2.6.30), it thus follows that the f-value given in (2.6.28) is asymptotically optimal for $0 < \psi \leq \frac{\lambda_2^2}{\lambda_1^2}$ or only if $\frac{\lambda_1^2}{\lambda_1^2 + 2\Delta^*\lambda_2^2} \leq \theta < 1$.

We thus conclude from our consideration of both of the above cases that for all θ under consideration, there is a unique asymptotically optimal f-value and a unique corresponding N-value.

2.7 Asymptotic Results for VT Sampling with Fixed Sample Size Termination Rule.

We may also employ the VT sampling rule with a fixed number of vectors, n. Let $S_{A,n}$ and $S_{B,n}$ be the number of successes with treatments A and B respectively, up to and including the n^{th} vector observation. Let $M_n = S_{A,n} - S_{B,n}$ and $M_n^* = S_{1,n} - S_{2,n}$. At termination, we accept one of three possible decisions:

(i) If $\frac{M_n}{n} > f$, then we make decision d_1.

(ii) If $-f \leq \frac{M_n}{n} \leq f$, then we make decision d_2.

(iii) If $\frac{M_n}{n} < -f$, then we make decision d_3.

Let $\overline{D}_1$ and $\overline{D}_2$ be defined similarly to D_1 and D_2 (defined at the beginning of the previous section) except that N is replaced by n. We choose n and f to satisfy the conditions:

$$(2.7.1) \qquad \text{(i)} \quad P\{\overline{D}_1\} \geq P_1^* \quad \text{if} \quad p_1 - p_2 \geq \Delta^*,$$

$$\text{(ii)} \quad P\{\overline{D}_2\} \geq P_2^* \quad \text{if} \quad p_1 = p_2.$$

It is clear that

$$(2.7.2) \qquad EM_n^* = n(p_1 - p_2); \ \text{Var}\ M_n^* = n[p_1(1-p_1) + p_2(1-p_2)]$$

and also that M_n^* is asymptotically normal with the above mean and variance. Let us first consider the left member of condition (i) of (2.7.1). We have from (2.7.2) that

$$(2.7.3) \qquad P\{\overline{D}_1\} = P\{M_n^* > nf\} = P\{W_n > - \frac{p_1 - p_2 - f}{\sqrt{p_1(1-p_1) + p_2(1-p_2)}} \sqrt{n}\}$$

$$\geq P\{W_n > -t_1(p_1, p_2)\} \sim \Phi(t_1(p_1, p_2)),$$

where

$$(2.7.4) \qquad t_1(p_1, p_2) = \frac{\Delta^* - f}{\sqrt{p_1(1-p_1) + p_2(1-p_2)}} \sqrt{n}$$

with $p_1 - p_2 = \Delta^*$ and where

$$(2.7.5) \qquad W_n = \frac{M_n^* - EM_n^*}{\sqrt{\operatorname{Var} M_n^*}} .$$

We find the pair (p_1, p_2) with $p_1 - p_2 \geq \Delta^*$ which minimizes $P\{\overline{D}_1\}$ for large n or equivalently which maximizes

$$(2.7.6) \qquad H_1 = \frac{n}{[t_1(p_1, p_2)]^2} = \frac{p_1(1-p_1) + p_2(1-p_2)}{[\Delta^* - f]^2}$$

with $p_1 - p_2 = \Delta^*$. As in (2.4.6), we easily find that $(\frac{1 - \Delta^*}{2}, \frac{1 + \Delta^*}{2})$ is the configuration of (p_1, p_2) that maximizes H_1 with maximum value, correct up to order $\mathcal{O}(\Delta^{*2})$, given by

$$(2.7.7) \qquad H_1^{\max} = \frac{1 - (\Delta^*)^2}{2(\Delta^* - f)^2} \approx \frac{1}{2(\Delta^* - f)^2} .$$

We therefore have the result that for $n \to \infty$, $\Delta^* \to 0$ and pairs (p_1, p_2) with $p_1 - p_2 \geq \Delta^*$,

$$(2.7.8) \qquad P\{\overline{D}_1\} \geq P_{LF(\overline{D}_1)}\{\overline{D}_1\} \sim \min_{(p_1, p_2)} \Phi\left(\sqrt{\frac{n}{H_1}}\right)$$

$$= \Phi((\Delta^* - f)\sqrt{2n}),$$

where the "$LF(\overline{D}_1)$" configuration is $(\frac{1 - \Delta^*}{2}, \frac{1 + \Delta^*}{2})$.

We now consider the left side of condition (ii) of (2.7.1). Clearly

$$(2.7.9)\quad P\{\overline{D}_2 | p_1 = p_2\} = P\{-nf \leq M_n^* \leq nf | p_1 = p_2\}$$

$$\sim 2\Phi\left(f\sqrt{\frac{n}{2p_1(1-p_1)}}\right) - 1.$$

The maximum value of $2p_1(1-p_1)$ occurs at $p_1 = \frac{1}{2}$ and is equal to $\frac{1}{2}$. We therefore conclude that for large n

$$(2.7.10)\quad \Phi\left(f\sqrt{\frac{n}{2p_1(1-p_1)}}\right) \geq \Phi(f\sqrt{2n}).$$

From (2.7.8), (2.7.9) and (2.7.10), the asymptotically $(N \to \infty)$ optimal pair (N, f) is found by solving the equations

$$(2.7.11)\quad \Phi((\Delta^* - f)\sqrt{N}) = P_1^*,$$

$$\Phi(f\sqrt{N}) = \frac{P_2^* + 1}{2},$$

for N and f, where $N = 2n$ is the total number of observations. Hence, we have to solve

$$(2.7.12)\quad (\Delta^* - f)\sqrt{N} = \lambda_1,$$

$$f\sqrt{N} = \lambda_2,$$

yielding the solutions

$$(2.7.13)\quad f = \frac{\lambda_2\Delta^*}{\lambda_2 + \lambda_1},$$

$$N = \left(\frac{\lambda_2 + \lambda_1}{\Delta^*}\right)^2.$$

In the next section, we compare the PW and VT sampling rules in the context of the three-decision problem $(N \to \infty,\ \Delta^* \to 0)$, using (2.7.13) and the results of the previous section.

2.8 Comparison of the Two Procedures for the Three-Decision Problem.

Let N_{PW} be the number of observations necessary to satisfy (2.6.1) with PW sampling using the asymptotically optimal f-value. Let N_{VT} be the corresponding quantity for the VT sampling rule. We consider two cases:

Case A.

If $\frac{2\Delta^* \lambda_2^2}{\lambda_1^2} \leq \psi \leq \frac{1 - \Delta^*}{1 + \Delta^*}$ (or equivalently if $\frac{1 + \Delta^*}{2} \leq \theta \leq \frac{\lambda_1^2}{\lambda_1^2 + 2\Delta^* \lambda_2^2}$), we know from (2.6.24) and (2.7.13) that

$$N_{PW} = \frac{(\lambda_2 + \sqrt{\lambda_2^2 + \lambda_1^2 \psi}\,)^2}{(\Delta^*)^2 \psi}, \tag{2.8.1}$$

$$N_{VT} = \left(\frac{\lambda_2 + \lambda_1}{\Delta^*}\right)^2 .$$

For the range of θ-values $\left(\frac{1 + \Delta^*}{2} \leq \theta \leq \frac{\lambda_1^2}{\lambda_1^2 + 2\Delta^* \lambda_2^2}\right)$, we show that $N_{PW} \geq N_{VT}$. From (2.8.1), we obtain after algebraic manipulation that $N_{PW} \geq N_{VT}$ if and only if

$$\psi \leq 4\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 + 2\lambda_1}\right)^2 . \tag{2.8.2}$$

From the fact that $\theta = \frac{1}{1 + \psi}$, the inequality (2.8.2) is equivalent to

$$\theta \geq \frac{1}{1 + g(\lambda_1, \lambda_2)}, \tag{2.8.3}$$

where

$$g(\lambda_1, \lambda_2) = 4\left(\frac{\lambda_2 + \lambda_1}{\lambda_2 + 2\lambda_1}\right)^2 . \tag{2.8.4}$$

It is easily seen that $g(\lambda_1, \lambda_2) \geq 1$ and hence $\frac{1}{1 + g(\lambda_1, \lambda_2)}$ is at most one half for any pair (P_1^*, P_2^*) with $P_i^* > \frac{1}{2}$ $(i = 1, 2)$, i.e., for any pair under consideration (see (2.6.2)). Hence it is sufficient to show that $\theta \geq \frac{1}{2}$ in order to show that $N_{PW} \geq N_{VT}$. Since $\theta \geq \frac{1+\Delta^*}{2}$ by assumption, this result holds and the proof for case A is completed.

Case B.

Suppose $0 < \psi \leq 2\Delta^*(\frac{\lambda_2}{\lambda_1})^2$ so that $\frac{\lambda_1^2}{\lambda_1^2 + 2\Delta^*\lambda_2^2} \leq \theta < 1$, then the expression for N_{VT} remains the same as in (2.8.1) but we have a different result [from (2.6.28)] for N_{PW} given by

$$(2.8.5) \qquad N_{PW} = \frac{1}{\psi}\left[\frac{\lambda_2(2-\Delta^*) + \lambda_1\sqrt{2\psi\Delta^*}}{\Delta^*}\right]^2 .$$

We prove that

$$(2.8.6) \qquad N_{PW} \geq N_{VT}$$

for this case $(0 < \psi \leq 2\Delta^* \frac{\lambda_2^2}{\lambda_1^2})$ in three steps. Step 1 is to show that N_{PW} is a decreasing function of ψ (or, equivalently, an increasing function of θ). [We expected N_{PW} to increase with θ, since an increase in θ imposes, by (2.6.1), a more stringent condition for our procedure to satisfy.] Step 2 is to note that at the dividing point given by $\psi = 2\Delta^*(\frac{\lambda_2}{\lambda_1})^2$ (or $\theta = \frac{\lambda_1^2}{\lambda_1^2 + 2\Delta^*\lambda_2^2}$), the value of N_{PW} given in Case A is less than the corresponding value given in Case B [see (2.6.24) and (2.6.28)]. [The difference in the value of N_{PW} at the dividing point between the two cases arises because the asymptotically optimal value of f in both cases is determined correct only up to order $\mathcal{O}(\Delta^{*2})$.] In step 3, the proof is completed by observing that we have already proved that $N_{PW} \geq N_{VT}$ for Case A, where we note that N_{VT} is independent of ψ.

We prove step 1 by differentiating $\sqrt{N_{PW}}$ wrt $\sqrt{\psi}$, obtaining

$$(2.8.7) \qquad \frac{d(\sqrt{N_{PW}})}{d(\sqrt{\psi})} = \begin{cases} -\dfrac{\lambda_2(2-\Delta^*)}{\psi\Delta^*} & \text{for } 0 < \psi \le 2\Delta^* \dfrac{\lambda_2^2}{\lambda_1^2} \\ -\dfrac{\lambda_2[\lambda_2 + \sqrt{\lambda_2^2 + \lambda_1^2\psi}\,]}{\Delta^*\psi\sqrt{\lambda_2^2+\lambda_1^2\psi}} & \text{for } 2\Delta^*\dfrac{\lambda_2^2}{\lambda_1^2} \le \psi \le \dfrac{1-\Delta^*}{1+\Delta^*} \end{cases}$$

and observing that it is negative for all ψ. By the previous paragraph, this proves (2.8.6).

We conclude from the results of this section that for $N \to \infty$ and $\Delta^* \to 0$, $N_{PW} \ge N_{VT}$ for all θ under consideration, i.e., $\frac{1+\Delta^*}{2} \le \theta < 1$; the introduction of θ is a necessary restriction on our problem, since the PW sampling rule is not workable for the unrestricted problem in our present context. In other words, even though we restrict our problem, the VT is still to be preferred over the PW sampling rule; our interest being in minimizing the total number of observations with fixed sample size termination rule.

2.9 Computer Results for Chapter II.

In this section, we investigate many of the asymptotic results ($\Delta^* \to 0$, $r \to \infty$) of this chapter and compare it, with the aid of the computer, to exact results for fixed Δ^* and P^*. In this way, we ascertain the value of these asymptotic results for moderate values of Δ^* and for values of r corresponding to P^*-values, which are most often used in practise.

We first concern ourselves with those results relating to Sections 2.2-2.4. For fixed P^* and Δ^*, the smallest value of the total number of observations N, say N_s, satisfying (1.1.1) [for procedure R_N] can be obtained by using the recursive equations (2.2.2) at $(p_1, p_2) = (\frac{1}{2} + \frac{\Delta^*}{2}, \frac{1}{2} - \frac{\Delta^*}{2})$ [the LF configuration], the boundary conditions (2.2.3) and equation (2.2.8). By (2.3.20), this value differs by at most one from the value of $N = 2n$, determined by finding the smallest number of vector observations, n, satisfying (1.1.1) for procedure $\bar{R}_n$, as is done in Sobel and Huyett [12]; this can be done by using (2.3.1) at the LF configuration.

Define $EN_B = EN_B(p_1, p_2)$ to be the expected number of observations on the poorer treatment at termination, under the configuration (p_1, p_2). We can determine $EN_B(p_1, p_2)$ by defining

(2.9.1) $$L_{M,d} = E[N_B | S_{1M} - S_{2M} = d \text{ and the next observation is on treatment 1}],$$

$$T_{M,d} = E[N_B | S_{1M} - S_{2M} = d \text{ and the next observation is on treatment 2}],$$

and by solving the recursive equations

(2.9.2) $$L_{M,d} = p_1 L_{M-1,d+1} + q_1 T_{M-1,d} \quad ,$$

$$T_{M,d} = p_2 T_{M-1,d-1} + q_2 L_{M-1,d} + 1 \ ,$$

with boundary conditions

(2.9.3) $$L_{0,d} = T_{0,d} = 0,$$

for

(2.9.4) $$EN_B = \frac{1}{2}(L_{N,0} + T_{N,0}).$$

We now find N_{PW} at $\Delta^* = .1$ and $.2$, for several values of P^* and then we evaluate $EN_B(p_1, p_2)$ at the LF configuration, when N_{PW}

is the total number of observations. The above results are summarized in Table 1 below.

Table 1

Exact (Computer) Results for R_N giving N_{PW} and EN_B

	$\Delta^* = .1$		$\Delta^* = .2$	
P^*	N_{PW}	$EN_B(.55, .45)$	N_{PW}	$EN_B(.6, .4)$
.99	539	242.6	134	53.7
.975	383	172.4	95	38.1
.95	270	121.5	67	26.9
.90	164	73.8	41	16.5
.85	108	48.6	27	10.9
.80	71	32.0	18	7.3
.75	46	20.7	12	4.9

The asymptotic results obtained from (2.3.22) and (2.4.31) are quite accurate; the values differ from those in Table 1 on the average of 1 percent.

From Table 1 and (2.3.21), we see the non-negligible savings in the expected number of observations on the poorer treatment for procedure R_N over $\bar{R}_n$. For $\bar{R}_N$, $EN_B = N_{PW}/2$. This difference will be even greater if p_1 and p_2 are further apart than in the least favorable configuration.

Next, we concern ourselves with those computer results relating to Section 2.5. Let S_{1M} (resp., S_{2M}) denote the number of successes with treatment 1 (resp., 2) when there are M observations still to be taken to reach a total of N. Letting for $i = 1$ and 2,

$$(2.9.5)\qquad L_{M,d}^{(i)} = P\{D_i | S_{1M} - S_{2M} = d \text{ and the next observation is on treatment 1}\},$$

$$T_{M,d}^{(i)} = P\{D_i | S_{1M} - S_{2M} = d \text{ and the next observation is on treatment 2}\},$$

the recursive equations for PW sampling [with fixed sample size N and with decision space given before (2.6.1)] are

$$(2.9.6)\qquad L_{M,d}^{(i)} = p_1 L_{M-1,d+1}^{(i)} + q_1 T_{M-1,d}^{(i)} \quad (i = 1, 2),$$

$$T_{M,d}^{(i)} = p_2 T_{M-1,d-1}^{(i)} + q_2 L_{M-1,d}^{(i)} \quad (i = 1, 2).$$

The boundary conditions are given by

$$(2.9.7)\qquad L_{0,d}^{(1)} = T_{0,d}^{(1)} = \begin{cases} 1 & \text{if } d > Nf \\ 0 & \text{if } d \leq Nf \end{cases},$$

and by

$$(2.9.8)\qquad L_{0,d}^{(2)} = T_{0,d}^{(2)} = \begin{cases} 0 & \text{if } d > Nf \\ 1 & \text{if } -Nf \leq d \leq Nf \\ 0 & \text{if } d < -Nf \end{cases},$$

where

$$(2.9.9)\qquad P\{D_i\} = \frac{1}{2}(L_{N,0}^{(i)} + T_{N,0}^{(i)}).$$

In this manner, we can compute $P\{D_i\}$ for fixed θ [see (2.6.1)], P_1^*, P_2^*, Δ^* and f, at the $LF(D_i)$ configuration for $i = 1$ and 2; we then insure (2.6.1) by choosing the smallest value of N, denoted by N_{PW}, so that

(2.9.10) $\quad P_{LF(D_i)}\{D_i\} \geq P_i^*,$

for $i = 1, 2$. It is found (using the computer) that for $f \geq .09$, the $LF(D_1)$ configuration occurs at $(p_1, p_2) = (.2, 0)$ and for all possible f, the $LF(D_2)$ configuration occurs at $(p_1, p_2) = (\theta, \theta)$, which agrees with (2.6.12) and the material immediately before (2.6.20). For our present purposes, we find it unnecessary to calculate the $LF(D_1)$ configuration for $f < .09$ and computer expense excourages us not to do so. We consider $\Delta^* = .2$, $P_1^* = .8$, $P_2^* = .6$ at three different values of θ, which determine our three cases.

Case (i).

Let us first consider $\theta = .6$. We find N_{PW} for several values of f and summarize our computer results in Table 2.

Table 2

Computer Results for Three Decision Problem with $\theta = .6$

f	.0873	.0900
N_{PW}	143	143

If $f \geq .09091$ than $N_{PW} \geq 153$ and if $f \leq .085$ than $N_{PW} \geq 142$. These results seem to indicate that $f = .0873$, although not unique, is an optimal (or close to optimal) procedure. It is seen from the asymptotic results (2.6.24) that $f = .0873$ is the optimal procedure, with N_{PW} equal to 135. Thus, for this case, the asymptotic results approximate the exact ones fairly well.

Case (ii).

Next, we consider $\theta = \frac{5}{7} \approx .71$. We find N_{PW} for several values of f and summarize our computer results in Table 3 below.

Table 3

Computer Results for Three Decision Problem with $\theta = 5/7$

f	.09000	.09091	.09170	.09340
N_{PW}	213	213	213	213

If $f < .09$, then $N_{PW} \geq 212$ and if $f \geq .095$, then $N_{PW} \geq 217$. These results indicate that $f = .09091$ is an optimal procedure. It is seen from the asymptotic results (2.6.28) that $f = .09091$ is the optimal procedure with N_{PW}-value equal to 216. We thus see that for this case, the asymptotic results (2.6.28) clearly approximate the exact ones.

Case (iii).

Finally, we consider $\theta = \frac{121}{161} \sim .75$. We find $N_{PW} = 252$ at $f = .09244$. Whether or not, $f = .09244$ is optimal is not investigated. This result is in close agreement with the asymptotic result (2.6.28) which states that at $f = .09244$, $N_{PW} = 252$. (2.6.28), in addition, states that this f-value is optimal.

Finally, we concern ourselves with those computer results relating to Section 2.7. Let m be the number of vector observations to be taken to reach a total of n vector observations. Let S_{1m} (resp., S_{2m}) denote the number of successes with treatment 1 (resp., 2), when there are m vector observations still to be taken to reach n. Letting, for $i = 1$ and 2,

$$(2.9.11) \qquad t_{m,d}^{(i)} = P\{\overline{D}_i \mid S_{1m} - S_{2m} = d\},$$

the recursive relations for VT sampling (with a fixed number of vectors n and with decision space given before (2.7.1)) needed to evaluate

$P\{\overline{D}_i\}$ are given for $i = 1, 2$ by

$$(2.9.12) \quad t^{(i)}_{m,d} = p_1q_2t^{(i)}_{m-1,d+1} + (p_1p_2 + q_1q_2)t^{(i)}_{m-1,d} + p_2q_1t^{(i)}_{m-1,d-1}.$$

The boundary conditions are given by

$$(2.9.13) \quad t^{(1)}_{0,d} = \begin{cases} 1 & \text{if } d > nf \\ 0 & \text{if } d \leq nf \end{cases},$$

and by

$$(2.9.14) \quad t^{(2)}_{0,d} = \begin{cases} 0 & \text{if } d > nf \\ 1 & \text{if } -nf \leq d \leq nf \\ 0 & \text{if } d < -nf \end{cases},$$

The desired result is

$$(2.9.15) \quad P\{\overline{D}_i\} = t^{(i)}_{n,0} .$$

In this manner, we can compute for fixed P^*_1, P^*_2, Δ^* and f, $P\{\overline{D}_i\}$ at the $LF(\overline{D}_i)$ configuration for $i = 1$ and 2, where the $LF(\overline{D}_1)$ configuration occurs at $(p_1, p_2) = (\frac{1}{2} + \frac{\Delta^*}{2}, \frac{1}{2} - \frac{\Delta^*}{2})$ and the $LF(\overline{D}_2)$ configuration occurs at $(p_1, p_2) = (\frac{1}{2}, \frac{1}{2})$; we then insure (2.7.1) by choosing the smallest value of $N = 2n$, denoted by N_{VT}, so that

$$(2.9.16) \quad P_{LF(\overline{D}_i)}\{\overline{D}_i\} \geq P^*_i$$

for $i = 1, 2$.

We do the above for $P^*_1 = .8$, $P^*_2 = .6$, $\Delta^* = .2$ and different values of f, the results of which are summarized in Table 4 below.

Table 4

Computer Results for Three Decision Problem (Vector-at-a-Time)

f	.0750	.0875	.090625	.09375	.096875	.1000
N_{VT}	108	92	90	86	84	84

f	.103125	.10625	.109375	.1125	.115625	.11875	.1250
N_{VT}	84	84	84	84	84	84	108

If $f > .125$ or $f < .075$, we find that $N_{VT} \geq 108$. These results indicate that $f = .1$, although not unique, is an optimal procedure. It is seen from the asymptotic results (2.7.13) that $f = .1$ is the optimal procedure, with N_{VT}-value equal to 72, which is somewhat away from the corresponding N_{VT}-value in Table 4. These asymptotic results become more accurate as $r \to \infty$ and $\Delta^* \to 0$. An indication of this fact is that at $P_1^* = .9$, $P_2^* = .8$ and $f = .1$, the exact value of $N_{VT} = 168$ and the asymptotic value of $N_{VT} = 164$.

CHAPTER III

Optimality of the PW Rule Using Inverse Sampling with Respect to a Given Class of Procedures Denoted by $\mathcal{R}$

3.1 Introduction.

In [14], a procedure denoted by R_I was studied. This procedure employs the PW sampling rule (with the usual randomization at the beginning of the experiment); its termination rule is "inverse sampling," i.e., experimentation is stopped when any one treatment achieves r successes. In this chapter, we study a class of procedures, $\mathcal{R}$ (say), containing R_I. The class $\mathcal{R}$ consists of procedures which we denote by $R_I^{(s)}$ for $s = 1, 2, \ldots$. All these procedures have the same termination rule, i.e., inverse sampling. The sampling rule of $R_I^{(s)}$ is to switch from one treatment to the other upon the occurrence of s successive failures; otherwise, we keep the same treatment (let us note that $R_I^{(1)} \equiv R_I$). We show that, in a certain sense, the optimal procedure in $\mathcal{R}$ is R_I.

3.2 PCS Considerations for the Class $\mathcal{R}$.

In this section, the size r of procedure $R_I^{(s)}$ is determined so that $P\{CS|R_I^{(s)}\}$ satisfies (1.1.1).

Let S_1(resp., S_2) denote the current number of successes with treatment 1 (resp., 2). Let the vector $\underset{\sim}{T}$ be equal to $(r - S_1, r - S_2)$. We note that $r - S_1$ (resp., $r - S_2$) is the number of successes that 1 (resp., 2) needs, to be selected as the better treatment. Let A_0 (resp., B_0) denote the event that treatment 1 (resp., 2) is to be used for our next observation without any failures on the same treatment in the immediately preceding trial. For $i = 1, 2, \ldots, s-1$ let A_i (resp., B_i) denote the event that the previous i trials were on treatment 1 (resp., 2), all resulting in failures and the same treatment is to be used on the next observation. We define the probabilities $R_{m,n}$; $R^{(i)}_{m,n}$; $S_{m,n}$ and $S^{(i)}_{m,n}$ for $i = 1, 2, \ldots, s-1$ in the following manner:

$$(3.2.1) \qquad \begin{aligned} R_{m,n} &= P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and } A_0\}, \\ R^{(i)}_{m,n} &= P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and } A_i\}, \\ S_{m,n} &= P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and } B_0\}, \\ S^{(i)}_{m,n} &= P\{CS \mid \underset{\sim}{T} = (m, n) \text{ and } B_i\}. \end{aligned}$$

Letting $R^{(j)}_{m,n} = S_{m,n}$ and $S^{(j)}_{m,n} = R_{m,n}$ for $j = s$, enables us to describe procedure $R_I^{(s)}$ for $s > 1$ by the recursive equations

$$(3.2.2) \qquad \begin{aligned} R_{m,n} &= p_1 R_{m-1,n} + q_1 R^{(1)}_{m,n}, \\ R^{(1)}_{m,n} &= p_1 R_{m-1,n} + q_1 R^{(2)}_{m,n}, \\ &\;\;\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ R^{(s-1)}_{m,n} &= p_1 R_{m-1,n} + q_1 S_{m,n}; \\ S_{m,n} &= p_2 S_{m,n-1} + q_2 S^{(1)}_{m,n}, \\ S^{(1)}_{m,n} &= p_2 S_{m,n-1} + q_2 S^{(2)}_{m,n}, \\ &\;\;\vdots \qquad\quad \vdots \qquad\qquad \vdots \\ S^{(s-1)}_{m,n} &= p_2 S_{m,n-1} + q_2 R_{m,n}. \end{aligned}$$

Simplifying (3.2.2), we have for $s \geq 1$ that

$$(3.2.3) \qquad \begin{aligned} R_{m,n} &= P_1 R_{m-1,n} + Q_1 S_{m,n}, \\ S_{m,n} &= P_2 S_{m,n-1} + Q_2 R_{m,n}, \end{aligned}$$

where $P_i = 1 - q_i^s$ and $Q_i = q_i^s$ $(i = 1, 2)$; we also have the boundary conditions

$$R_{0,n} = 1, \; S_{m,0} = 0 \quad \text{for} \quad n, m \geq 1. \tag{3.2.4}$$

It is clear that the recursive equations (3.2.3) and the boundary conditions (3.2.4) apply when the PW sampling scheme (procedure R_I) is used with treatments 1 and 2 which have probability of success P_1 and P_2, respectively (instead of p_1 and p_2, as usual). Treatment 1, of course, is still the better treatment. Let us denote procedure R_I used in this special setup as $R_I(P_1, P_2)$ [Note that $R_I(p_1, p_2)$ is what we commonly refer to as R_I]. Let A_1 (resp., B_1) denote the event that treatment 1 (resp., 2) is used on the first observation. The event $\{CS|A_1\}$ [resp., $\{CS|B_1\}$] for PW sampling is equivalent to the event that experimentation ends with the r^{th} success by A, given that it begins with an observation on A (resp., B) and at termination, A and B have an equal number of failures [resp., B has one more failure than A] while B has not yet achieved its r^{th} success. The $P\{CS|A_1\}$ [resp., $P\{CS|B_1\}$] is then equal to the probability of the event that the number of failures until A achieves its r^{th} success is less than or equal to (resp., less than) the number of failures until the r^{th} success by B. Hence, from (3.2.2), (3.2.3) and (3.2.4),

$$\begin{aligned} P\{CS|R_I^{(s)}\} &= P\{CS|R_I(P_1, P_2)\} = \tfrac{1}{2} P\{CS|A_1\} + \tfrac{1}{2} P\{CS|B_1\} \\ &= P\{NB(r, P_1) < NB(r, P_2)\} + \tfrac{1}{2} P\{NB(r, P_1) = NB(r, P_2)\}, \end{aligned} \tag{3.2.5}$$

where $NB(a, b)$ is a negative binomial chance variable which represents the number of failures until the a^{th} success, with probability of success b on each trial. For $Y = NB(a, b)$, we have the probability law

$$(3.2.6) \qquad P\{Y = y|(a, b)\} = b^a(1-b)^y\binom{y+a-1}{y};\ y = 0, 1, \ldots .$$

The mean value and the variance of Y are given by

$$(3.2.7) \qquad EY = \frac{1-b}{b}\ , \ \mathrm{Var}\ Y = \frac{1-b}{b^2}\ .$$

By (3.2.5), we have from the central limit theorem that

$$(3.2.8) \qquad P\{CS|R_I^{(s)}\} \sim P\{W_r < (P_1 - P_2)\sqrt{\frac{r}{Q_1P_2^2 + Q_2P_1^2}}\}$$

$$= P\{W_r < \Delta\sqrt{\frac{r}{D}}\},$$

where

$$(3.2.9) \qquad W_r = \frac{[(NB(r, P_1) + r) - (NB(r, P_2) + r)] - r[\frac{1}{P_1} - \frac{1}{P_2}]}{\sqrt{r(\frac{Q_1}{P_1^2} + \frac{Q_2}{P_2^2})}}\ ,$$

$$(3.2.10) \qquad D = \frac{Q_1P_2^2 + Q_2P_1^2}{(q_2^{s-1} + q_1q_2^{s-2} + \ldots + q_1^{s-1})^2}\ .$$

We wish to minimize $P\{CS|R_I^{(s)}\}$ (for large r, subject to the condition that $\Delta \geq \Delta^*$) or equivalently (by (3.2.8)) to minimize $\Delta\sqrt{\frac{r}{D}}$. It is easy to see that for any (q_1, q_2) configuration such that $q_2 - q_1 \geq \Delta^*$, the $P\{CS\}$ can be reduced by decreasing q_2 to $q_1 + \Delta^*$. Our problem thus reduces to minimizing $\Delta\sqrt{\frac{r}{D}}$ over all (q_1, q_2) configurations such that $q_2 - q_1 = \Delta^*$. To accomplish this, we first set $\Delta = \Delta^*$ and then substitute $q_1 = q_0 - \frac{\Delta^*}{2}$, $q_2 = q_0 + \frac{\Delta^*}{2}$ into D and attempt to maximize the resulting expression, denoted by $D(q_0)$, over the values q_0 in the closed interval $[\frac{\Delta^*}{2}, 1 - \frac{\Delta^*}{2}]$. Three cases are considered:

Case (a).

We first consider the case where $s = 1$. It is shown in [14] that $D(q_0)$ is maximized at $q_0 = \frac{1}{3} + \Theta(\Delta^{*2})$ and the maximum value of D, D^{max} (say), is

$$(3.2.11) \qquad D^{max} = \frac{8}{27} + \Theta(\Delta^{*2}).$$

Case (b).

Next, we consider the case where $s = 2$. D is then given by

$$(3.2.12) \qquad D = \frac{q_1{}^2(1-q_2{}^2)^2 + q_2{}^2(1-q_1{}^2)^2}{(q_2 + q_1)^2} .$$

Upon simplification, we have that $D(q_0)$ is given by

$$(3.2.13) \qquad D(q_0) = \frac{1}{128q_0{}^2} [64q_0^6 - (128 + 16(\Delta^*)^2)q_0^4 + (64 + 64(\Delta^*)^2 - 4(\Delta^*)^4)q_0{}^2$$
$$+ 16(\Delta^*)^2 - 8(\Delta^*)^4 + (\Delta^*)^6] .$$

The derivative of $D(q_0)$ wrt q_0, $\frac{dD(q_0)}{dq_0}$, is

$$(3.2.14) \qquad \frac{dD(q_0)}{dq_0} = \frac{1}{128q_0^3} [256q_0^6 - (256 + 32(\Delta^*)^2)q_0^4 - 32(\Delta^*)^2$$
$$+ 16(\Delta^*)^4 - 2(\Delta^*)^6].$$

It is easy to see that $\frac{dD(q_0)}{dq_0}$ is negative for all q_0 between $\frac{\Delta^*}{2}$ and $1 - \frac{\Delta^*}{2}$; $D(q_0)$ is therefore maximized at $q_0 = \frac{\Delta^*}{2}$, i.e., we thus have that the least favorable (LF) configuration (the (q_1, q_2) configuration that minimizes the $P\{CS\}$ over all pairs (q_1, q_2) with $q_2 - q_1 \geq \Delta^*$) occurs for large r at $(q_1, q_2) = (0, \Delta^*)$.

Case (c).

It is conjectured that for the case $s \geq 3$, the maximum value of $D(q_0)$, as in the case for $s = 2$, occurs at $q_0 = \frac{\Delta^*}{2}$. We conclude by finding, for large r, the smallest value of r so that procedure $R_I^{(s)}$ $[s \geq 1]$ satisfies (1.1.1); two cases are considered.

Case 1.

Let us first consider the case when $s = 1$. By (3.2.8), our discussion for case (a), and the central limit theorem, it follows that, for large r and small Δ^*,

$$P\{CS|R_I^{(1)}\} \geq P_{LF}\{CS|R_1^{(1)}\} \sim \Phi(\Delta^*\sqrt{\frac{r}{D^{max}}}), \tag{3.2.15}$$

where $P_{LF}\{CS|R_I^{(1)}\}$ is the probability of correct selection evaluated for the least favorable configuration. Thus, asymptotically, to find the smallest value of r, r_1 (say), so that (1.1.1) is satisfied for procedure $R_I^{(1)}$, we set $\Phi(\Delta^*\sqrt{\frac{r}{D^{max}}}) = P^*$ and solve for r. We thus have that (for large r and small Δ^*)

$$r_1 \sim \frac{8}{27}(\frac{\lambda}{\Delta^*})^2, \tag{3.2.16}$$

where λ is a constant chosen so that $\Phi(\lambda) = P^*$.

Case 2.

Next, we consider the case when $s \geq 2$. An approximation to the r-value of interest using the technique of Case 1, provides a poor approximation in this case. Quite an accurate approximation [see tables 5, 6], though, is obtained by using the fact that the LF configuration occurs at $(p_1, p_2) = (1, 1 - \Delta^*)$, for $s \geq 2$. To do this, a result of Feller is used (see [4], page 325). Suppose, we observe n independent Bernoulli trials; on each trial, either a success (S) or a failure (F) is observed, with probabilities h and $1 - h$, respectively. A sequence of n letters S and F contains as many success runs of length r as there are non-overlapping uninterrupted successions of exactly r letters S; a success of length r occurs at the j^{th} trial if the j^{th} trial adds a new run to the sequence. Thus in SSS/SF/SSS/SSS/, we have three success runs of length 3, and they occur at trials number 3, 8,11; there are five success runs of length 2 and they occur at trials number 2, 4, 7, 9, 11. Failure runs of length r can be defined analogously. Feller proves that the probability that there aren't any success runs of length t in n trials is given by

$$(3.2.17) \qquad \ell_n \sim \frac{1 - hx}{(t+1-tx)(1-h)} \cdot \frac{1}{x^{n+1}},$$

where x is the unique positive root of the polynomial $1 - (1-h)y[1 + hy + \ldots + h^{t-1}y^{t-1}]$. We make use of this result in the next paragraph.

The least favorable configuration for procedure $R_I^{(s)}$ $(s \geq 2)$ [cases (b) and (c)] occurs at $(p_1, p_2) = (1, 1 - \Delta^*)$. Under this configuration, an incorrect selection (IS) occurs only if treatment 2 achieves r successes, with no observations being made on treatment 1. Since it takes approximately $\frac{r}{1 - \Delta^*}$ observations on 2 to attain r successes (on 2) (the number of trials until r successes being distributed like the rv $[r + NB(r, 1 - \Delta^*)]$ {see (3.2.6)}, having mean value $\frac{r}{1 - \Delta^*}$), the $P\{IS|R_I^{(s)}\}$ $(s \geq 2)$ is approximately equal to one half multiplied by the probability of no failure runs of length s in $\frac{r}{1 - \Delta^*}$ trials, where the probability of a failure on any trial is Δ^*. Let us note that the one half in the previous sentence results from the necessity that we begin experimentation with treatment 2. Thus, using (3.2.17), with $h = \Delta^*$, $t = s$ and $n = \frac{r}{1 - \Delta^*}$, it follows that

$$(3.2.18) \qquad P\{IS|R_I^{(s)}\} \sim \frac{1}{2}\left(\frac{1 - \Delta^* x_s}{(s+1-sx_s)(1-\Delta^*)}\right) \cdot \frac{1}{x_s^{(r/1-\Delta^*)+1}}$$

for $s \geq 2$, where x_s is the unique positive root of the polynomial $1 - (1-\Delta^*)y[1 + \Delta^* y + \ldots + (\Delta^*)^{s-1}y^{s-1}]$. To satisfy (1.1.1), for large r, we set

$$(3.2.19) \qquad P\{IS|R_I^{(s)}\} = 1 - P^*$$

and solve for r, resulting in the solution, denoted by $r_s (s \geq 2)$,

$$(3.2.20) \qquad r_s \sim (1-\Delta^*)\left[\frac{\log\{\frac{1 - \Delta^* x_s}{(s+1-sx_s)(1-\Delta^*)}\} - \log 2(1-P^*)}{\log x_s} - 1\right].$$

It is useful, for the next section, to prove the result that for fixed P^* close to 1,

$$(3.2.21) \qquad \lim_{\Delta^* \to 0} \frac{r_s}{r_1} = c(P^*, \Delta) > 1$$

for $s \geq 2$, where $c(P^*, s)$ is a constant (possibly $+\infty$), which can depend on P^* and s.

Let us first prove that for P^* close to 1,

$$(3.2.22) \quad \lim_{\Delta^* \to 0} \frac{r_2}{r_1} = c(P^*, 2) \geq \frac{27}{8} \frac{\log 2(1-P^*)}{2\log(1-P^*)} > 1.$$

Helpful in proving this is the fact that for large P^* and small Δ^*,

$$(3.2.23) \quad r_1 \leq \frac{8}{27} \left[\frac{-2 \log(1-P^*)}{(\Delta^*)^2}\right],$$

which follows from another result of Feller (see [4], page 175) stating that as $x \to \infty$

$$(3.2.24) \quad 1 - \Phi(x) \sim \frac{1}{(2\pi)^{1/2} x} e^{-\frac{1}{2}x^2},$$

where Φ is the standard normal cdf. Applying (3.2.24) to the λ in (3.2.16), we have that

$$(3.2.25) \quad 1 - P^* = 1 - \Phi(\lambda) \sim \frac{1}{(2\pi)^{1/2}\lambda} e^{-\frac{1}{2}\lambda^2},$$

and thus (3.2.23) has been proved. Also from (3.2.18), $x_2 = 1 + (\Delta^*)^2 + o(\Delta^{*2})$ [e.g., for $\Delta^* = .1$, $x_s = 1.00925$] and thus

$$(3.2.26) \quad r_2 \approx \frac{-\log 2(1-P^*)}{(\Delta^*)^2}$$

for Δ^* small; the fact that $\log(1 + (\Delta^*)^2) \approx (\Delta^*)^2$ is used in (3.2.26). The result (3.2.22) then follows from (3.2.23) and (3.2.26). Since from (3.2.18), x_s is decreasing with s, we can similarly prove (3.2.21).

[It is conjectured that

$$(3.2.27) \quad x_s = 1 + O(\Delta^{*s}),$$

which would imply that

$$(3.2.28) \quad \lim_{\Delta^* \to 0} \frac{r_{s+1}}{r_s} = \infty$$

for $s \geq 2$. An indication of (3.2.27) is that for $\Delta^* = .1$, $x_3 = 1.00090$. If true, (3.2.27) would yield a nice result, discussed in the next section.]

We make use of the results of this section, after a discussion of the expected number of observations necessary to reach a decision for the class of procedures $\mathcal{R}$.

3.3 Optimality of the PW-rule in the Class $\mathcal{R}$ Using the EN Criterion.

Let N denote the number of observations until a decision is reached as to the better treatment. In this section, we obtain the expected value of N, EN, for the class $\mathcal{R}$ and show that it is minimized for procedure R_I when Δ^* is small.

We define the expected number of additional observations required $R_{m,n}$; $R^{(i)}_{m,n}$; $S_{m,n}$ and $S^{(i)}_{m,n}$ for $i = 1, 2, \ldots, s-1$ in the following manner

$$(3.3.1) \quad \begin{aligned} R_{m,n} &= E[N|\underset{\sim}{T} = (m,n) \text{ and } A_0], \\ R^{(i)}_{m,n} &= E[N|\underset{\sim}{T} = (m,n) \text{ and } A_i], \\ S_{m,n} &= E[N|\underset{\sim}{T} = (m,n) \text{ and } B_0], \\ S^{(i)}_{m,n} &= E[N|\underset{\sim}{T} = (m,n) \text{ and } B_i]. \end{aligned}$$

where A_0, A_i, B_0, B_i are defined on page 51.

For $s > 1$ we can write the recursive equations for EN in the form

$$(3.3.2)\qquad \begin{aligned} R_{m,n} &= p_1 R_{m-1,n} + q_1 R^{(1)}_{m,n} + 1; & S_{m,n} &= p_2 S_{m,n-1} + q_2 S^{(1)}_{m,n} + 1; \\ R^{(1)}_{m,n} &= p_1 R_{m-1,n} + q_1 R^{(2)}_{m,n} + 1; & S^{(1)}_{m,n} &= p_2 S_{m,n-1} + q_2 S^{(2)}_{m,n} + 1; \\ &\ \vdots & &\ \vdots \\ R^{(s-2)}_{m,n} &= p_1 R_{m-1,n} + q_1 R^{(s-1)}_{m,n} + 1; & S^{(s-2)}_{m,n} &= p_2 S_{m,n-1} + q_2 S^{(s-1)}_{m,n} + 1; \\ R^{(s-1)}_{m,n} &= p_1 R_{m-1,n} + q_1 S_{m,n} + 1; & S^{(s-1)}_{m,n} &= p_2 S_{m,n-1} + q_2 R_{m,n} + 1. \end{aligned}$$

Simplifying (3.3.2), we have for $s \geq 1$ that

$$(3.3.3)\qquad \begin{aligned} R_{m,n} &= P_1 R_{m-1,n} + Q_1 S_{m,n} + \frac{P_1}{p_1}, \\ S_{m,n} &= P_2 S_{m,n-1} + Q_2 R_{m,n} + \frac{P_2}{p_2}, \end{aligned}$$

where P_1, P_2, Q_1, Q_2 are defined after (3.2.3). We also have the boundary conditions

$$(3.3.4)\qquad R_{0,n} = 0,\ S_{m,0} = 0 \quad \text{for all } n \geq 1 \text{ and all } m \geq 1.$$

We note that EN is given by

$$(3.3.5)\qquad EN = \tfrac{1}{2}[R_{r,r} + S_{r,r}].$$

We define generating functions U and V by

$$(3.3.6)\qquad U = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} R_{m,n} x^m y^n,\ V = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} S_{m,n} x^m y^n.$$

Multiplying both sides of the two equations in (3.3.3) by $x^m y^n$, summing over (m, n), and using (3.3.4) yields

$$U = P_1 xU + Q_1 V + \frac{P_1}{p_1} \frac{x}{1-x} \frac{y}{1-y}, \tag{3.3.7}$$

$$V = P_2 yV + Q_2 U + \frac{P_2}{p_2} \frac{x}{1-x} \frac{y}{1-y}.$$

Solving for V, we obtain

$$V = [\frac{P_2}{p_2}(1 - P_1 x) + Q_2 \frac{P_1}{p_1}] \frac{xy}{(1-x)(1-y)} D, \tag{3.3.8}$$

where D is given by

$$D = \frac{1}{(1-P_2 y)(1-P_1 x) - Q_1 Q_2} = \sum_{i=0}^{\infty} \frac{(Q_1 Q_2)^i}{[(1-P_2 y)(1-P_1 x)]^{i+1}} \tag{3.3.9}$$

$$= \sum_{i=0}^{\infty} (Q_1 Q_2)^i \sum_{j=0}^{\infty} \binom{i+j}{j} P_1^j x^j \sum_{k=0}^{\infty} \binom{i+k}{k} P_2^k y^k.$$

From (3.3.7), (3.3.8) and (3.3.9), the coefficient of the $x^j y^k (j \geq 1, k \geq 1)$ term of U and V, denoted by $u_{j,k}$ and $v_{j,k}$, respectively, are given by

$$u_{j,k} = \sum_{i=0}^{\infty} (Q_1 Q_2)^i [Q_1 \frac{P_2}{p_2} \sum_{\ell=0}^{j-1} \binom{i+\ell}{\ell} P_1^\ell + Q_1 Q_2 \frac{P_1}{p_1} \sum_{\ell=0}^{j-1} \binom{i+1+\ell}{\ell} P_1^\ell] \tag{3.3.10}$$

$$\cdot \sum_{t=0}^{k-1} \binom{i+t}{t} P_2^t + \frac{P_1}{p_1} \frac{1 - P_1^j}{Q_1},$$

$$v_{j,k} = \sum_{i=0}^{\infty} (Q_1 Q_2)^i [\frac{P_2}{p_2} \sum_{\ell=0}^{j-1} \binom{i-1+\ell}{\ell} P_1^\ell + Q_2 \frac{P_1}{p_1} \sum_{\ell=0}^{j-1} \binom{i+\ell}{\ell} P_1^\ell]$$

$$\cdot \sum_{t=0}^{k-1} \binom{i+t}{t} P_2^t,$$

where we define $\sum_{\ell=0}^{j-1} \binom{\ell-1}{\ell} P_1^{\ell} = 1$. Using (3.3.10) and the well known identity for the incomplete Beta function (see e.g. page 4 of [14])

$$(3.3.11) \qquad I_{Q_1}(i+1, r) = Q_1^{(i+1)} \sum_{\ell=0}^{r-1} \binom{i+\ell}{\ell} P_1^{\ell} .$$

We readily find from (3.3.5)

$$(3.3.12) \qquad E[N|R_I^{(s)}] = \frac{1}{2} \{ \sum_{i=0}^{\infty} I_{Q_2}(i+1, r) [(\frac{P_1}{P_1Q_1} + \frac{P_2}{P_2Q_2}) I_{Q_1}(i+1, r)$$

$$+ \frac{P_1}{Q_1P_1} I_{Q_1}(i+2, r) + \frac{P_2}{Q_2P_2} I_{Q_1}(i, r)] + \frac{P_1}{P_1} \frac{1 - P_1^r}{Q_1} \}.$$

It is easy to see from (3.3.11) that

$$(3.3.13) \qquad I_{Q_1}(i+1, r) = P_1^r \sum_{\ell=i+1}^{\infty} \binom{r-1+\ell}{\ell} Q_1^{\ell},$$

by noting that

$$(3.3.14) \qquad I_{P_1}(r, i+1) = 1 - I_{Q_1}(i+1, r).$$

Defining $E_r[I_{P_2}(r, X)]$ as

$$(3.3.15) \qquad E_r[I_{P_2}(r, X)] = P_1^r \sum_{\ell=0}^{\infty} \binom{\ell+r-1}{\ell} Q_1^{\ell} I_{P_2}(r, \ell)$$

and using the relation (see page 6 of [14])

$$(3.3.16) \qquad \sum_{i=0}^{\infty} I_{Q_1}(i+1, r) I_{Q_2}(i+1, r) = r \frac{Q_1}{P_1} - \frac{r}{P_1} E_{r+1}[I_{P_2}(r, X)]$$

$$+ \frac{r}{P_2} E_r[I_{P_2}(r+1, X)]$$

and the relations (from (3.3.16))

$$(3.3.17) \qquad \sum_{i=0}^{\infty} I_{Q_1}(i, r) I_{Q_2}(i+1, r) = \sum_{i=0}^{\infty} I_{Q_1}(i+1, r) I_{Q_2}(i+1, r)$$

$$+ 1 - E_r[I_{P_2}(r, X+1)],$$

$$(3.3.21) \qquad \frac{E[N|R_I^{(1)}]}{E[N|R_I^{(2)}]} < \frac{27}{16}\frac{r_1}{r_2} .$$

Using (3.2.22) and noting that $\lim\limits_{x \downarrow 0} \frac{\log 2x}{2\log x} = \frac{1}{2}$, this implies that

$$(3.3.22) \qquad E[N|R_I^{(1)}] < E[N|R_I^{(2)}],$$

for $P^* \to 1$, $\Delta^* \to 0$ and all configurations (p_1, p_2) with $p_1 \geq p_2$.

[For $P^* \to 1$, $\Delta^* \to 0$ and all pairs (p_1, p_2) with $p_1 \geq p_2$ it is conjectured that $E[N|R_I^{(s)}]$ is an increasing function of s for all s. The inequality (3.3.22) and conjecture (3.2.27) provide a sufficient condition for this to be true.]

3.4 Computer Results for Chapter III.

In this section, we concern ourselves with exact results, determined with the aid of the computer, relating to Chapter III.

For fixed Δ^* and P^*, the recursive equations (3.2.3), the boundary conditions (3.2.4) and the fact that $P\{CS\} = \frac{1}{2}(R_{r,r} + S_{r,r})$ can be used first to find the LF configurations of procedures $R_I^{(s)}$ to any required degree of accuracy, and next to find the smallest value of r such that (1.1.1) is satisfied for procedure $R_I^{(s)}$. Corresponding to the value of $r = r_s$, the expected number of observations until termination, $E_sN(p_1, p_2)$, under the configuration (p_1, p_2) and using procedure $R_I^{(s)}$, can then be obtained, using the recursive equations (3.3.3), the boundary conditions (3.3.4) and equation (3.3.5). We do the above for $s = 1$ and 2, $\Delta^* = .1$ and several values of P^*, and we evaluate $E_sN(.5, .4)$; which is not at the LF configuration. Our results are summarized in Table 5 below.

Table 5

Exact (Computer) Results for $R_I^{(s)}$ for $s = 1$ and 2

P^*	r_1	$E_1N(.5, .4)$	r_2	$E_2N(.5, .4)$
.99	161	591	390	1360
.975	114	418	299	1043
.95	81	297	230	803
.90	49	179	161	563
.85	33	119	120	419
.80	22	78	92	321
.75	14	48	69	241

Computer expense prevents us from doing the same for $s = 3$; however we do find that at $\Delta^* = .1$ and $P^* = .6$, $r_3 = 224$ and $E_3(.5, .4) = 743$. For $\Delta^* = .1$, the LF configurations for procedures $R_I^{(1)}$ and $R_I^{(s)}[s = 2, 3]$ are found to be $(p_1, p_2) = (.72, .62)$ and $(p_1, p_2) = (1, .9)$, accurate to the nearest $.01^{th}$. This agrees with our asymptotic $(\Delta^* \to 0, r \to \infty)$ LF configurations of $(\frac{2}{3} + \frac{\Delta^*}{2}, \frac{2}{3} - \frac{\Delta^*}{2})$ and $(1, 1 - \Delta^*)$, respectively.

Let us note that the results in Table 5 are closely approximated by our asymptotic results (3.2.16), (3.2.20) and (3.3.21). Using these asymptotic results, we write up Table 6, similar to Table 5. Also, at $P^* = .6$, the asymptotic results are $r_3 = 225$ and $E_3N(.5, .4) = 741$, which agree with the corresponding exact results in Table 5.

Table 6

Asymptotic Results for $R_I^{(s)}$ for s = 1, 2 and 3 based on (3.2.16) and (3.3.21)

P^*	r_1	$E_1N(.5, .4)$	r_2	$E_2N(.5, .4)$	r_3	$E_3N(.5, .4)$
.99	161	590	383	1334	3912	12895
.975	114	417	294	1022	2996	9876
.95	80	292	226	786	2303	7592
.90	49	178	158	551	1611	5308
.85	32	118	119	413	1205	3973
.80	21	77	90	315	918	3025
.75	13	49	69	239	695	2289

CHAPTER IV

Two Open Questions

4.1 Question One.

The two open questions in this area of research, that were referred to at the end of Section 1.2, are now considered. In this section, the first of these questions is discussed.

Suppose the conditions of our experiment dictate that we sample in blocks of size two, with each block containing two observations on the same treatment. At the beginning of the experiment, the treatment to be used on the first block of observations, is determined by randomization. Sampling is terminated the first time any treatment (either treatment A or B) achieves r successes and we declare that treatment to be the better one. We consider a class of procedures consisting of members denoted by R_h, where $0 \leq h \leq 1$. Procedure R_h is to (1) retain the same treatment after a double success, (2) switch treatments after a double failure and (3) retain the same treatment with probability h after a single success and failure (in the block of size two). Our intention is to compare the performance of these procedures.

Define the probabilities $R_{m,n}$ and $S_{m,n}$ by

$$(4.1.1)\qquad \begin{aligned} R_{m,n} &= P\{CS \mid r - S_1 = m,\ r - S_2 = n \text{ and the next block of observations is on treatment } 1\}, \\ S_{m,n} &= P\{CS \mid r - S_1 = m,\ r - S_2 = n \text{ and the next block of observations is on treatment } 2\}, \end{aligned}$$

where $S_1(S_2)$ is the number of successes from the better (worse) treatment.

The probability of correct selection for procedure R_h can then be obtained using the recursive equations

$$(4.1.2) \qquad R_{m,n} = p_1^2 R_{m-2,n} + 2p_1q_1[hR_{m-1,n} + (1-h)S_{m-1,n}] + q_1^2 S_{m,n},$$

$$S_{m,n} = p_2^2 S_{m,n-2} + 2p_2q_2[hS_{m,n-1} + (1-h)R_{m,n-1}] + q_2^2 R_{m,n},$$

with boundary conditions (for $m, n \geq 1$)

$$(4.1.3) \qquad S_{0,n} = R_{0,n} = R_{-1,n} = 1,$$

$$R_{m,0} = S_{m,0} = S_{m,-1} = 0.$$

The probability of correct selection is then given by

$$(4.1.4) \qquad P\{CS\} = \tfrac{1}{2}(R_{r,r} + S_{r,r}).$$

In a similar manner, we can obtain $E_hN(p_1, p_2)$, the expected number of observations until termination, using procedure R_h, under the configuration (p_1, p_2), by defining the expectations

(4.1.5) $R_{m,n} = E[N|r - S_1 = m, r - S_2 = n$ and the next block of observations is on treatment 1],

$S_{m,n} = E[N|r - S_1 = m, r - S_2 = n$ and the next block of observations is on treatment 2],

and then using the recursive equations

$$(4.1.6) \qquad R_{m,n} = p_1^2 R_{m-2,n} + 2p_1q_1[hR_{m-1,n} + (1-h)S_{m-1,n}] + q_1^2 S_{m,n} + 1,$$

$$S_{m,n} = p_2^2 S_{m,n-2} + 2p_2q_2[hS_{m,n-1} + (1-h)R_{m,n-1}] + q_2^2 R_{m,n} + 1,$$

with boundary conditions (for $m, n \geq 1$)

(4.1.7) $$S_{0,n} = R_{0,n} = R_{-1,n} = 0,$$

$$R_{m,0} = S_{m,0} = S_{m,-1} = 0.$$

The expected number of observations is then given by

(4.1.8) $$EN = \frac{1}{2}(R_{r,r} + S_{r,r}).$$

First, setting $\Delta^* = .1$ and using (4.1.2), (4.1.3) and (4.1.4), we obtain the LF configuration for procedure R_h, when $h = 0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}$ and 1, and in turn obtain (for each procedure) the smallest value of r, denoted by r_h, so that (1.1.1) is satisfied, at different values of P^*. Next, using (4.1.6), (4.1.7) and (4.1.8), we obtain $E_h(.5, .4)$ for the above values of h, corresponding to $r = r_h$. These results are summarized in Table 7.

Table 7

Computer Results for Procedure R_H

	R_0	$R_{1/4}$	$R_{1/2}$	$R_{3/4}$	R_1
LF	(.72,.62)	(.74,.64)	(.78,.68)	(.85,.75)	(1,.9)

P^*	r_0	$r_{1/4}$	$r_{1/2}$	$r_{3/4}$	r_1
.99	176	203	239	298	708
.975	125	144	170	212	542
.95	89	102	120	150	417
.90	54	62	74	92	291
.85	36	41	49	61	218
.80	24	28	33	41	166
.75	16	18	21	27	125

Table 7 (cont.)

P^*	$E_0N(.5,.4)$	$E_{1/4}N(.5,.4)$	$E_{1/2}N(.5,.4)$	$E_{3/4}N(.5,.4)$	$E_1N(.5,.4)$
.99	334	380	439	532	1201
.975	237	269	312	379	920
.95	169	191	221	268	708
.90	102	116	136	164	495
.85	67	76	89	109	371
.80	44	51	60	72	283
.75	29	32	37	47	213

These results (obtained with the aid of a computer) indicate the possibility that $E_hN(p_1, p_2)$ is minimized at $h = 0$. Satisfactory analytic results, however, have been obtained only for procedure R_1. For fixed $\Delta^* \to 0$, we obtain the result that for procedure R_1 the LF configuration occurs at

(4.1.9) $$(p_1, p_2) = (1, 1 - \Delta^*)$$

and that

(4.1.10) $$r_1 \approx \frac{2\log 2(1-P^*)}{(1+\Delta^*)\log[1-(\Delta^*)^2]} .$$

Also, for fixed $r \to \infty$,

(4.1.11) $$E_1N(p_1, p_2) \approx \frac{r}{2} [\frac{q_2^2 + q_1^2}{(1-q_1)q_2^2}].$$

These results are in agreement with the computer results in Table 7 for procedure R_1.

It is easily seen that this problem can be generalized to blocks of size greater than two and also to problems in which all of the observations within the same block do not have to be made on the same treatment.

4.2 Question Two.

In the past, we have represented the performance of a treatment on a single observation by a binomial random variable, which takes the values 0 and 1 according to whether the treatment achieves a failure or a success. We now suppose that a treatment can achieve one of three levels of performance on a single observation. We rate its performance by the numbers 0, 1 and 2, where the higher number corresponds to the better performance. The assumption is made that for a given treatment, the probability of achieving a certain level of performance does not change from one observation to the next. We are still concerned with the problem of finding the "better" of two treatments, denoted by treatment 1, from the results of a series of independent observations, made on both treatments A and B.

Let the random variable X (resp., Y) which takes the values 0, 1 and 2, with probabilities q_1, q_2, q_3 (resp., q_1', q_2', q_3') represent the performance of treatment 1 (resp., 2) on a single observation. We assume that treatment 1 is the "better" treatment in the sense that

$$(4.2.1) \qquad q_3 = P\{X = 2\} \geq P\{Y = 2\} = q_3',$$

$$q_2 + q_3 = P\{X = 1 \text{ or } 2\} \geq P\{Y = 1 \text{ or } 2\} = q_2' + q_3'.$$

It is easily seen that (4.2.1) is equivalent to

$$(4.2.2) \qquad q_1 \le q_1' , \quad q_1 + q_2 \le q_1' + q_2' ,$$

i.e., X is stochastically larger than Y. [We recall that a random variable U is stochastically larger than the random variable V if $P\{U \le t\} \le P\{V \le t\}$, for all real t.] The two inequalities in (4.2.1) can also be put in the simple form

$$(4.2.3) \qquad q_1 \le q_1' , \quad q_3 \ge q_3' .$$

As a consequence of (4.2.1),

$$(4.2.4) \qquad EX = q_2 + 2q_3 \ge q_2' + 2q_3' = EY.$$

Only those procedures, R, are considered, which satisfy the requirement that

$$(4.2.5) \qquad P\{CS|R\} \ge P^*,$$

whenever $q_1' - q_1 \ge \Delta^*$ and $q_3 - q_3' \ge \Delta^*$. This requirement is analogous to requirement (1.1.1) for the binomial problem. We call the present problem "the trinomial problem," in contrast to the binomial problem of Sobel and Weiss.

We now consider a class of sampling rules, under a termination rule corresponding to inverse sampling for the trinomial problem.

Experimentation is begun by randomizing (with equal probability) among the two treatments. The number assigned to a given treatment on a single observation is to be called the number of points scored by the treatment on that observation. Experimentation is stopped as soon as one treatment scores r points and we declare that treatment to be the better one. We consider a class of procedures, consisting of members denoted by R_h', where $0 \leq h \leq 1$. Procedure R_h' is to (1) retain the same treatment after a score of two points, (2) switch treatments after a score of no points, and (3) retain the same treatment with probability h, after a score of one point. Our intention is to compare, in a meaningful manner, these R_h' procedures.

Letting S_1 (resp., S_2) be the current total score of treatment 1 (resp., 2), we define the probabilities $R_{m,n}$ and $S_{m,n}$ as

$$(4.2.6) \qquad \begin{aligned} R_{m,n} &= P\{CS \mid r - S_1 = m,\ r - S_2 = n \text{ and the next observation is on treatment } 1\}, \\ S_{m,n} &= P\{CS \mid r - S_1 = m,\ r - S_2 = n \text{ and the next observation is on treatment } 2\}. \end{aligned}$$

The probability of correct selection for procedure R_h' can then be obtained by using the recursive equations

$$(4.2.7) \qquad \begin{aligned} R_{m,n} &= q_3 R_{m-2,n} + q_2[hR_{m-1,n} + (1-h)S_{m-1,n}] + q_1 S_{m,n}, \\ S_{m,n} &= q_3' S_{m,n-2} + q_2'[hS_{m,n-1} + (1-h)R_{m,n-1}] + q_1' R_{m,n}, \end{aligned}$$

with boundary conditions (for $m, n \geq 1$)

(4.2.8) $$S_{0,n} = R_{0,n} = R_{-1,n} = 1 ,$$

$$R_{m,0} = S_{m,0} = S_{m,-1} = 0 .$$

The probability of correct selection is then given by

(4.2.9) $$P\{CS\} = \frac{1}{2}(R_{r,r}+ S_{r,r}).$$

In a similar manner, we can obtain $E_hN(q_1, q_3, q_1', q_3')$, the expected number of observations until termination, using procedure R_h', under the configuration (q_1, q_3, q_1', q_3'), by defining the expectations

(4.2.10) $$R_{m,n} = E[N|r - S_1 = m,\ r - S_2 = n \text{ and the next observation is on treatment } 1],$$

$$S_{m,n} = E[N|r - S_1 = m,\ r - S_2 = n \text{ and the next observation is on treatment } 2],$$

and then using the recursive equations

(4.2.11) $$R_{m,n} = q_3R_{m-2,n}+ q_2[hR_{m-1,n}+ (1-h)S_{m-1,n}] + q_1S_{m,n}+ 1,$$

$$S_{m,n} = q_3'S_{m,n-2}+ q_2'[hS_{m,n-1}+ (1-h)R_{m,n-1}] + q_1'R_{m,n}+ 1,$$

with boundary conditions (for $m, n \geq 1$)

(4.2.12) $$S_{0,n} = R_{0,n} = R_{-1,n} = 0,$$

$$R_{m,0} = S_{m,0} = S_{m,-1} = 0.$$

The expected number of observations is then given by

(4.2.13) $$EN = \frac{1}{2}(R_{r,r}+ S_{r,r}).$$

First, setting $\Delta^* = .1$ and using (4.2.7), (4.2.8) and (4.2.9), we find that the LF configuration for procedure R_h', when $h = 0$, $\frac{1}{2}$ and 1, occurs at a point where $q_2 = q_2' = 0$. This implies that the three procedures have the same LF configuration; since the recursive equations (4.2.7) are then independent of h. Specifically, the LF configuration for these three procedures occurs at $(q_1, q_3, q_1', q_3') = (.28, .72, .38, .62)$. Next, for different values of P^*, we obtain the smallest value of r, denoted by r', so that (4.2.5) is satisfied for procedure R_0'. Since the LF configuration occurs at a point where $q_2 = q_2' = 0$, r' is also the smallest value of r so that (4.2.5) is satisfied for procedures $R_{1/2}$ and R_1. Finally, using (4.2.11), (4.2.12) and (4.2.13), we obtain $E_hN(.4, .5, .5, .4)$ for the above values of h, corresponding to $r = r'$. These results (obtained with the aid of a computer) are summarized in Table 8.

Table 8

Computer Results for Procedure R_h'

P^*	r'	$E_hN(.4,.5,.5,.4)$		
		h = 0	h = 1/2	h = 1
.99	321	536	532	527
.975	227	379	376	373
.95	161	269	267	264
.90	97	161	160	159
.85	65	107	106	105
.80	43	70	69	69
.75	27	43	43	42

These results indicate the possibility that $E_h N(q_1, q_3, q_1', q_3')$ is minimized at $h = 1$. Satisfactory analytic results, however, have been obtained only for procedure R_1'. For fixed $\Delta^* \to 0$, we obtain the result that the LF configuration for procedure R_1' occurs at

$$(4.2.14) \quad (q_1, q_3, q_1', q_3') = (\frac{1}{3} - \frac{\Delta^*}{2}, \frac{2}{3} + \frac{\Delta^*}{2}, \frac{1}{3} + \frac{\Delta^*}{2}, \frac{2}{3} - \frac{\Delta^*}{2})$$

and that

$$(4.2.15) \quad r' \approx \frac{\lambda^2}{(\Delta^*)^2} \frac{16}{27},$$

where λ is such that $\Phi(\lambda) = P^*$. Also, for fixed $r \to \infty$,

$$(4.2.16) \quad E_1 N(q_1, q_3, q_1', q_3') \approx \frac{r}{1 - q_1 + q_3} [\frac{q_1' + q_1}{q_1'}] .$$

These results are in agreement with the computer results (in Table 8) for procedure R_1'.

The usual Sobel and Weiss formulation for the binomial problem and our present setup for the trinomial problem have in common the fact that the random variable associated with the performance of treatment 1 on a single observation (say, X) is stochastically larger than the corresponding random variable for treatment 2 (say, Y). With this in mind, it is easy to see that the present problem can be extended to the more general multinomial situation. It is safe to conjecture that a further extension of this problem to cases when the random variables X and Y are more generally distributed (even, possibly continuous), as long as X is stochastically larger than Y, would also prove fruitful.

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